

# A DENOTATIONAL ENGINEERING OF PROGRAMMING LANGUAGES

To make software systems reliable  
and user manuals clear, complete, and consistent

*A book in statu nascendi  
(a working version)*

Andrzej Jacek Blikle  
Piotr Chrzastowski-Wachtel  
Janusz Jabłonowski  
Andrzej Tarlecki

*It always seems impossible  
until it's done.*

Nelson Mandela

Warsaw, May 26<sup>th</sup>, 2025



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# 1 INTRODUCTION

## 1.1 What motivated us to write this book?

It is a well-established engineering practice that designing a new product starts from a blueprint supported by mathematical calculations. Both provide a mathematical warranty that the future functionality of the product will satisfy the expectations of its designer and user.

In software engineering, the situation is different. In the place of blueprints and calculations, programmers develop their codes starting from a contract between a future user and an IT producer, usually written in a more or less technical language but without mathematical rigor. The future target code is developed from such a contract through a sequence of steps, where this contract is “translated” to increasingly more and more technical descriptions and ends up with a compilable code. Although all these descriptions have a professional character, they still do not offer a mathematical precision comparable to, e.g., differential equations of a bridge designer.

As a consequence, large parts of budgets for program developments are spent on testing, i.e., removing errors introduced at the coding stage. Since testing may only discover some faults but never guarantee their absence, the non-discovered bugs are passed on to users to be removed later under the name “maintenance”. In several cases, this situation led to spectacular catastrophes practically always — to many nuisances for users. The latter are, therefore, forced to accept a disclaimer like the following one:

*There is no warranty for the program to the extent permitted by applicable law. Except when otherwise stated in writing, the copyright holders and/or other parties provide the program "as is" without warranty of any kind, either expressed or implied, including, but not limited to, the implied warranties of merchantability and fitness for a particular purpose. The entire risk as to the quality and performance of the program is with you. Should the program prove defective, you assume the cost of all necessary servicing, repair, or correction.*

Is it possible that a producer of a car, a dishwasher, or a building could request such a disclaimer from their clients? Why, then, is the software industry an exception? In our opinion, one of the causes of this situation is a lack of adequate mathematical tools for software engineers to guarantee the functional reliability of their products.

Of course, we are aware that mathematical tools will not solve all problems of software engineering, as they are not solving all problems in other industries. At the same time, however, we are convinced that there is a need for “more mathematics” in software production. This book proposes two sets of mathematical tools for software engineers: one addressed to language designers and the other to programmers. We believe that taking responsibility by software engineers for their products should be possible to the same extent as in remaining industries.

## 1.2 Building mathematically correct programs

The issue of mathematically provable program correctness appeared for the first time in a work by Alan Turing [93] published in conference proceedings *On High-Speed Calculating Machines*, which took place at Cambridge University in 1949. Later, that subject was investigated for several decades under the name of “proving program correctness”, but the developed methods never became a standard tool for the software industry. Consequently, many people concluded that research in this field is not worth the effort. A particularly explicit formulation of this opinion we found in a monograph, *Deductive Software Verification* [2], published in 2016:

*For a long time, the term formal verification was almost synonymous with functional verification. In the last years, it became more and more clear that full functional verification is an elusive goal for almost all application scenarios. Ironically, this happened because of advances in verification technology: with the advent of verifiers, such as KeY, that mostly cover and precisely model industrial languages and that can handle realistic systems,*

*it finally became obvious just how difficult and time-consuming the specification of the functionality of real systems is. Not verification, but specification is the real bottleneck in functional verification.*

We strongly believe that the failure to prove program correctness in practical scenarios has two primary sources.

The first is an obvious observation that proofs of theorems are usually longer than the theorems themselves. Therefore, proofs of program correctness may contain thousands, if not millions of lines, which makes “hand-made proofs” unrealistic. Additionally, fully formalized proofs for “practical” programs are hardly possible due to the lack of formal semantics of the languages in which they have been written.

The second cause is even more critical — programs that are supposed to be proven correct are usually incorrect! Consequently, correctness proofs are regarded as methods of identifying errors in programs. Besides, the order, first a program and then the proof of its correctness, may seem natural for mathematicians — first a hypothesis and then its proof — but is somewhat awkward for engineers, who first prepare blueprints and calculations and only then build “their bridges”.

To our knowledge, the inadequacy of the approach of first writing a program and only then trying to prove its correctness was pointed out for the first time by Edsger Dijkstra in 1976 in his book *A Discipline of Programming* [51] where he writes:

*Between the lines (of his book) the reader may have caught a few more general messages. The first message is that it does not suffice to design a mechanism of which we hope that it will meet its requirements, but we must design it in such a form that we can convince ourselves — and anyone else for that matter — that it will, indeed, meet its requirements. And, therefore, instead of first designing a program and then trying to prove its correctness, we develop correctness proof and program hand in hand (our emphasis). (In actual fact, the correctness proof is developed slightly ahead of the program: after having chosen the form of the correctness proof, make the program so that it satisfies the proof's requirements.)*

Dijkstra formalized this idea using his *weakest preconditions*. A little later (1977 – 1981), Andrzej Blikle published a few papers [25] - [28] technically different from Dijkstra's approach but in a similar spirit. In recent years, Dijkstra's ideas have been implemented by the authors of Dafny Environment [69] and their followers (cf. Tabea Bordis et al. [41]) under the name correct-by-construction.

In Dijkstra's approaches, the authors tacitly assume that their proposed program constructors are sound. They do not prove this soundness since their languages lack mathematical semantics. In our approach, we first show how to build programming languages with fully mathematical semantics and then how to develop sound program-construction rules for such languages.

### 1.3 Designing languages with mathematical semantics

By a mathematical semantics of a programming language, we shall mean a function that assigns meanings to programs. Since the 1970s, many researchers have started to believe that such semantics, to be “practical,” must be compositional, i.e., that the meaning of a whole must be a composition of the meanings of its parts. Later, such semantics were called *denotational* — the meaning of a program is its *denotation* — and for about two decades, researchers investigated the possibilities of defining denotational semantics for existing programming languages. The two most complete (although not fully formalized) such semantics were written in 1980 for Ada [16] and CHILL [42] in a metalanguage VDM [15]. Later, in 1987, Andrzej Blikle described a denotational semantics of a subset of Pascal [29]. In the latter case, the metalanguage was MetaSoft [29], primarily based on VDM.

Unfortunately, none of these attempts created a denotational semantics of a widely used programming language. In our opinion, this situation was caused by the fact that for most language designers, the *meaning of a program* is a compiler's behavior generated by the execution of this program. Also, programmers, when they explain to each other what a given piece of syntax “means”, describe what a compiler will “do” during the execution of this syntax. This machine-oriented understanding of program meanings has led to syntaxes that are not “suitable” for giving them denotational semantics. Technical arguments supporting this opinion will be offered later in Sec. 6 and 7.



Independent of these problems, many researchers have found denotational models technically too complicated. Let us quote just one such opinion expressed by Cliff Johns in his book *Understanding Programming Languages* [63]:

*Denotational semantics is mathematically elegant but requires some fairly sophisticated mathematical concepts in order to describe programming languages of the sort that are used to build real applications.*

Such opinions about denotational semantics were associated with their early-stage technicalities. One was a *jump instruction* **goto**, and the other — self-applicable procedures that can take themselves as parameters (Algol 60, see [7]). The former had led to *continuations* [89], the latter to *reflexive domains* [88]. Continuations were counterintuitive and reflexive domains — mathematically fairly complicated. Fortunately, although these mechanisms were considered necessary in the 1960s, they were abandoned ten years later.

In our approach, we use neither continuations nor reflexive domains. The idea of denotational semantics without these mechanisms was described by Andrzej Blikle and Andrzej Tarlecki in a joint paper [39] in 1983.

Besides the resignation of “historical” technicalities of denotational semantics, we use in our approach an idea proposed by Andrzej Blikle in [30] that in developing a programming language, we should start from its denotations **DEN** and derive from them a syntax **SYN** later. He proved that, in this case, a denotational semantics

$$\mathbf{DS} : \mathbf{SYN} \rightarrow \mathbf{DEN}$$

always exists and, additionally, is unique. Formally, **SYN** and **DEN** constitute many-sorted algebras (Sec. 2.12), and the associated semantics **DS** is a (unique) homomorphism between them. As it turns out, there is a simple method — to a large extent algorithmizable — of deriving syntax from (the description of) denotations and, later, the semantics for both of them.

To illustrate our method, we designed a virtual programming language, **Lingua**. At the level of data structures, it includes booleans, integers, reals, texts, records, arrays, and their arbitrary combinations plus objects. It is equipped with a relatively strong mechanism of user-definable datatypes on the one hand and object types, i.e., classes, on the other. Control structures available in **Lingua** include structural instructions and multi-recursion procedures. The language has a complete error-reporting mechanism, by which we mean that every run-time error (except infinite looping) is signaled by an error message. Of course, errors will not occur in correct programs.

At the end of the book, we show how to enrich **Lingua** by two following mechanisms:

1. an API for SQL,
2. concurrency at the level of simple Petri nets,

Of course, **Lingua** is not developed/described in all detail since, in such a case, the book would hardly be readable. Our exercise with **Lingua** only illustrates a language-designing method that (hopefully) may be used in some future to design and implement practical programming languages.

Nevertheless, an experimental interpreter of **Lingua** has been developed by a group of students attending courses given commonly by Andrzej Blikle and Alex Schubert at the Department of Mathematics, Informatics, and Mechanics of Warsaw University in the academic years 2019/20 and 2020/21.

Once we have a language with denotational semantics, we can define program-construction rules and prove their soundness. Our construction rules were sketched for the first time in [25]. In the present book, they are developed for **Lingua**. Technically, the rules are used to build so-called *metaprograms* that syntactically include their specifications. Program construction rules guarantee that if we combine two or more correct programs into a new program or transform a correct program, we get a correct one again. Therefore, the correctness proofs of programs are implicit in how they are developed and in the soundness proofs of construction rules.

## 1.4 What is in the book?

As mentioned in the preceding sections, the book contains many thoughts developed during 1960-1990 but later abandoned. One of the teams developing these ideas was working at the Institute of Computer Science of the Polish Academy of Sciences, and two of us — A.Blikle and A.Tarlecki — enjoyed working there. We then



created a semi-formal metalanguage called **MetaSoft** [29] dedicated to formal definitions of programming languages.

Sec.2 introduces general mathematical tools used later in describing our basic model. In particular, they include:

1. a formal, but not formalized, definition of **MetaSoft**,
2. fixed-point theory in partially ordered sets,
3. the calculus of binary relations,
4. formal-language theory,
5. fixed-point domain equations based on so-called *naive denotational semantics* (rather than Scott and Strachey's reflexive domains),
6. many-sorted algebras,
7. abstract errors as tools for the description of error-handling mechanisms,
8. three-valued predicate calculi of McCarthy and Kleene,
9. equational grammars (equivalent to Chomsky's grammars and BNF's),
10. syntactical algebras based on equational grammars,
11. a short half-formal reminder of LL(k) grammars.

It should be emphasized in this place that Sec.2 may discourage less mathematically oriented readers. These readers may skip reading this section, maybe except Sec. 2.1 where notational conventions are explained. All mathematical tools and facts mentioned there are necessary to prove the mathematical soundness of our approach but are not prerequisites to understanding it.

Sec. 3 includes an intuitive description of our model and defines significant milestones to be passed through in its construction.

Sec. 4 is devoted to developing a general model of data structures and their types. Types, which are frequently regarded as sets of data, in our model are finitistic structures that indicate the "shapes" of the corresponding data. This approach allows for a definition of an algebra of types at a data level and an algebra of the denotations of type expressions at the level of denotations.

In Sec. 5 we introduce three fundamental concepts: a class, an object, and a state. We also discuss the visibility (privacy) issues of objects' attributes, typical of object-oriented mechanisms.

Sec. 6 describes the core of our model and is devoted to denotations. Here, we define the carriers and the constructors of an algebra of denotations. From a practical viewpoint, when we design an algebra of denotations, we make significant decisions on the tools offered by a language to programmers.

Once denotations are defined, we proceed in Sec. 7 to the derivation of syntax. Here, given a description of the algebra of denotations, we derive step-by-step three syntaxes: an abstract syntax, a concrete syntax, and a colloquial syntax. Formally, these syntaxes constitute algebras and are described by equational grammars. We also show how to derive a formal definition of a function of semantics once we are given an algebra of denotations and "its" algebra of syntax.

Sec.8 is devoted to an abstract theory of partial and total correctness of programs. This theory bases on an algebra of binary relations, making it universal for many programming languages.

In Sec. 9, starting from (already defined) **Lingua**, we develop a corresponding language for *validated programming*, **Lingua-V**. In this language, programs are developed in a way that guarantees their (provable) correctness. However, we do not build a logic of programs — like C.A.R. Hoare [61], [5] or E. Dijkstra [50], [51] did — but we define a list of (proved) rules for the construction of provably correct programs. The correctness proofs of such programs are implicit in the sequence of rules that have been used in developing them. Such programs will be said to be *proved by construction*. In this place, it is worth mentioning a similar approach to program development in the Dafny project (see [41], [68], [69] and [70]). The authors of this project refer to their programs as *correct by construction*. The difference with our approach is that **Lingua** is given mathematical (denotational) semantics, and our program-construction rules are proved to be sound (preserving correctness), whereas the language of Dafny has only intuitive semantics, and consequently, their construction rules have solely intuitive justification.

Sec. Sec. 10 and 11 are devoted to enriching **Lingua** with SQL mechanisms. Since we do not expect our reader to be familiar with SQL details, we provide in Sec. 10 an intuitive introduction to its primary tools. In Sec 11, we give these tools a denotational model. As it turns out, extending **Lingua** to **Lingua-SQL** is more than just a simple enrichment of a source algebra of denotations by new carriers and constructors. It requires profound modifications and, therefore, offers a non-trivial example of enhancing a denotational model by new mechanisms.

Even more substantial changes to our model are necessary when, in Sec. 12, we introduce some concurrency mechanisms into **Lingua**. In this case, the flowchart-like structures of our programs are replaced by simple Petri nets enriched by the trace languages of Mazurkiewicz [76].

In Sec. 13, we included short remarks about what remains to be done in our project. In particular, we are talking about two computer-assisted work environments: one for the designers of languages using our method and another for programmers in such languages.

## 1.5 What differentiates our approach from the others?

Historically, the ideas of denotational engineering emerged from the early works of A. Blikle ([18] to [33]), A.Blikle with A. Mazurkiewicz [37], and A.Blikle with A. Tarlecki [39]. In turn, these works followed various approaches in this or another way. Below, we give references to the earliest papers on these approaches and to the significant contributions that followed:

- generative grammars of N. Chomsky ([44] in 1956, [45] in 1957, [46] in 1959, [47] in 1962, [55] in 1966, and [17] in 1971),
- denotational semantics of D. Scott's and Ch. Strachey's ([88] in 1971 and [89] in 1977),
- C.A.R Hoare's logic of programs (the founding paper [61] in 1969, and surveys [4] in 1981, [5] in 2020 and [6] in 2020),
- E. Dijkstra's total correctness of programs and the derivation of correct programs ([50] in 1968 and [51] in 1976),
- many-sorted algebras in computer science by J. A Goguen, J.W, Thatcher, E. G. Wagner and J. B. Wright ([58] in 1977), and M. Wirsing [96]
- three-valued propositional calculus of J. McCarthy (cf. [78] in 1967),
- abstract errors in program's semantics originally introduced by Joe Goguen ([57] in 1978, [28] in 1981, [11] in 1984, [29] in 1987 and [31] in 1988), and later also investigated from a perspective of a Hoare's logic by J. V. Tucker and J. I. Zucker ([92] in 1988).

The main differences between our approach and other approaches to denotational semantics and program correctness are the following:

1. In the field of programming language design:
  - 1.1. our denotational models are based on set theory rather than D. Scott's and Ch. Strachey reflexive domains,
  - 1.2. the denotations of programs are state-to-state functions rather than continuation-to-continuation,
  - 1.3. denotations are developed in the first place, and syntax is derived from them later; the process of the derivation of syntax is highly algorithmizable,
  - 1.4. the idea of a colloquial syntax allows making syntax user-friendly without damaging mathematical rigor,
  - 1.5. our denotational models include:
    - 1.5.1. error-detection mechanisms supported by three-valued boolean expressions and predicates,
    - 1.5.2. objects and classes,
    - 1.5.3. SQL databases,
    - 1.5.4. simple Petri nets concurrency.
2. In the field of correct program development:

- 2.1. the soundness of program construction rules is proved on the grounds of a denotation semantics of the involved language,
- 2.2. the use of three-valued predicates enriches Dijkstra's total correctness approach by a clean-termination property.

## 1.6 Where are we on the way from user expectations to an executable code?

A virtual production line in a software factory may be seen — from a simplified perspective — as a sequence of the following actions:

1. the identification of user's needs (either in a dialog with the user or by market research),
2. the creation of a technical vision of future software architecture; this step usually includes many substeps where our intuitive image of the future software is gradually concretized,
3. the creation of a high-level program (coding) expected to be an adequate algorithmization of the output of 2.,
4. a compilation process,
5. running the compiled code by hardware.

Of course, each of these stages offers many error opportunities. Why, then, have we restricted our attention to step 3.?

The first answer is that our research experience predisposes us to tackle high-level program development more than the other steps. Besides, there is quite a lot of mathematical research available in this field (cf. Sec. 1.5).

Our second answer is based on the fact that compilers and hardware are much better tested today — due to their extensive use — than applications that are just being created, and therefore our choice of 3. before 4. and 5. seems partly justified.

Why then 3., rather than 1. or 2.? In this case, in addition to our former arguments, we can say that we do not know of languages used at early stages of software development that could be given mathematical semantics. At the same time, however, *domain-specific languages* may offer a promising perspective. Therefore, it may be vital to see to what extent our language development technique may open research opportunities in this area.

## 2 METASOFT AND ITS MATHEMATICS

From 1970 to 1990, Andrzej Blikle had been lecturing mathematical foundations of computer science to IT practitioners. In these cases, he frequently heard an objection that there is too much mathematics that software engineers have to swallow. Bosses of IT departments expected that their teams could be “trained” in that new mathematics within one weekend, well maximally two. In such cases, he tried to bring to their attention that future mechanical or electrical engineers attend two to five semesters of mathematics during their university studies. However, most of this mathematics was created at the borderline between the XIX and XX century and is oriented towards physics, astronomy, and classical engineering rather than informatics.

At the beginning of the second half of the XX century, mathematicians started to think about mathematical theories for computer science; some of the branches of mathematics earlier considered “unpractical” — such as set theory, mathematical logic, or abstract algebras — became their standard tools. A little later, new branches emerged: theory of abstract automata and formal languages, logic of programs, models of concurrent systems, and many others. Today, mathematical foundations of computer science embrace large and still fast-growing new branches of applied mathematics.

In the present section, we describe selected mathematical tools we shall use in the book. At the same time, we are conscious that going through this section may be pretty challenging for some readers. We may advise them only to slip over this math and possibly return to it when some technique used in subsequent sections will require a deeper justification.

### 2.1 Basic notational conventions of MetaSoft

#### 2.1.1 General rules

**MetaSoft** is a semi-formal (i.e., not fully formalized) mathematical notation used in describing denotational models of programming languages. Each such model consist of three mathematical entities:

1. *Denotations* — the meanings of programs and their components such as expressions, instructions, declarations etc.
2. *Syntaxes* — programs and their components.
3. *Semantics* — a function that assigns denotations to syntaxes.

In the colloquial English of computer scientists denotations are most frequently confused with semantic. We can hear, e.g., that “the semantics of instructions are functions that modify memory states”. In our approach we shall strictly distinguish between *denotations* that are the meanings of programs, *syntaxes* that are strings of characters used by programmers, and *semantics* that are function mapping *syntaxes* into *denotation*. We shall describe this fact by the following formula:

$$\text{SEM} : \text{Syntaxes} \mapsto \text{Denotations}$$

The notation used in this formula will be explained in Sec. 2.1.3. So far we may only notice that our mathematical formulas will be typeset in *Arial*, rather than in *Times New Roman Italic* as usual in mathematical texts. The reason of this decision is twofold:

1. in our texts we want to distinguish between an informal layer typeset in Times New Roman including its italic versions, and the layer of formulas,
2. as we are going to see, large and complex formulas that we shall use are better readable in Arial than in *Times Italic*.

In turn, to carefully distinguish between syntax and denotations, programs and their components will be typeset in **Arial Narrow**, e.g.,

**while**  $x > 100$  **do**  $x := x-1$  **od**

Additionally, since **while**, **do** and **od** are keywords, they are typeset in bold.

Another special property of **MetaSoft** is that (meta)mathematical variables that denote elements, sets and functions are frequently many-character symbols rather than single letters. This choice has been forced by the fact that in denotational models we use many more symbols than in “usual” mathematics, and therefore we should by giving them mnemotechnical forms. A typical example of a MetaSoft formula such as

$\text{ind} : \text{InsDen} = \text{WfState} \rightarrow \text{WfState}$

is read as follows: the domain of instruction denotations **InsDen** is the set of partial functions that transform well-formed states into well-formed states. Elements of this domain will be denoted by **ind** possibly with prefixes or postfixes.

Another special notation concerns indexed variables. In traditional mathematics indices are written as subscripts, e.g., as  $a_i$ . Since this complicates typing and is not compatible with the syntaxes used in programs, we shall frequently write indices at the same level as an indexed symbol, e.g., as  $a\text{-}i$ . Of course, indices may be many-character symbols as well.

Our special notational conventions have one more justification. As we are going to see, the descriptions of denotational models in **MetaSoft** resemble programs, in particular codes of interpreters. In the future the writing of such descriptions should be assisted by dedicated editors. The outputs of these editors will then become inputs for generators of interpreters or compilers of corresponding programming languages.

Logical operators are given mnemotechnical names: **and**, **or**, **not**, **tt**, **ff**. The two last are logical constants “true” and “false”. For quantifiers we shall use:

$\forall$  — *general quantifier* (for all)

$\exists$  — *existential quantifier* (there exists)

Instead of  $i = 1, \dots, n$  we shall write  $i = 1;n$ . By “iff” we shall mean “if and only if”, and by “wrt” — “with respect to”.

## 2.1.2 Sets

Symbol  $\{\}$  denotes an empty set and

$\{\text{ele-1}, \dots, \text{ele-n}\}$  or  $\{\text{ele-i} \mid i = 1;n\}$

denote finite sets of  $n$  elements. The fact that **ele** is, or is not, an element of a set of elements **Element** we shall write as

$\text{ele} : \text{Element}$  or respectively as  $\text{ele} /: \text{Element}$

For any sets **A** and **B** their inclusion will be written as

$A \subseteq B$

By

$A \mid B$  and  $A \cap B$

we denote the union and the intersection of these sets. If **FamSet** is a family of sets then

$\bigcup \text{FamSet}$

denotes the union of all the sets of this family. By

$$A \times B$$

w denote the Cartesian product of sets. The expression:

$$A \times B \times C \times D$$

denotes the set of tuples of the form  $(a, b, c, d)$ , whereas the expression:

$$A \times (B \times C) \times D$$

denotes the set of tuples of the form  $(a, (b, c), d)$ , and analogously for other combinations of parentheses. For every  $n \geq 0$  the  $n$ -th Cartesian power  $A^{cn}$  of a set  $A$  is the set of all  $n$ -tuples of the elements of  $A$ , i.e.:

$$A^{c0} = \{()\} \quad \text{— the only element of that set is an empty tuple}$$

$$A^{cn} = \{(a_1, \dots, a_n) \mid a_i \in A\} \text{— for } n > 0$$

Given Cartesian powers, we can define two other operations:

$$A^{c+} = \bigcup \{A^{cn} \mid n > 0\} \quad \text{— Cartesian plus operation,}$$

$$A^{c*} = A^{c0} \mid A^{c+} \quad \text{— Cartesian star operation.}$$

The set of all subsets of  $A$  and respectively of all finite subsets of  $A$  is denoted by

$$\text{Sub}.A$$

$$\text{FinSub}.A$$

The following notations shall be used for sets of relations and functions:

$$\text{Rel.}(A, B) \quad \text{— the set of all binary relations between } A \text{ and } B; \text{ i.e., the set of all subsets of } A \times B; \text{ more about binary relations in Sec.2.7,}$$

$$A \rightarrow B \quad \text{— the set of all } \textit{partial functions} \text{ from } A \text{ to } B, \text{ i.e., functions that do not need to be defined for all elements of } A,$$

$$A \mapsto B \quad \text{— the set of all } \textit{total functions} \text{ from } A \text{ to } B, \text{ i.e., functions that are defined for all elements of } A; \text{ notice that each total function is a partial function but not vice-versa,}$$

$$A \Rightarrow B \quad \text{— the set of all } \textit{mappings} \text{ from } A \text{ to } B, \text{ i.e., functions defined for only a finite subset of } A.$$

Following this notation by

$$f : A \rightarrow B$$

we mean that  $f$  is an element of the set  $A \rightarrow B$ , i.e. is a partial function from  $A$  to  $B$ , and analogously for other operators creating sets of functions.  $A$  is called the *domain* of  $f$ , and  $B$  is called its *range*. The use of colon “:” also explains why the traditional  $a \in A$  we write as  $a : A$ .

### 2.1.3 Functions

For practical reasons, the value of a function  $\text{fun}$  for argument  $a$  shall be written as  $\text{fun}.a$  rather than  $\text{fun}(a)$ . Why this is practical will be seen a little later. The expression

$$\text{fun}.a = ? \tag{2.1-1}$$

means that  $\text{fun}$  is not defined for  $a$ . It does not mean that “?” is anything like an “undefined element”. The expression  $\text{fun}.a = ?$  stands for

$$\text{not } (\exists b)(\text{fun}.a=b)$$

Analogously

$\text{fun.a} = !$

stands for  $(\exists b)(\text{fun.a}=b)$ . For an arbitrary function

$\text{fun}: A \rightarrow B$

and an arbitrary set  $C$  by the *truncation of* function  $f$  to  $C$  we shall mean:

$\text{fun truncate-to } C = \{(a, \text{fun.a}) \mid a : A \cap C\}$ .

The *domain of definedness* of function  $f$  is the set where  $f$  is defined, i.e.

$\text{dom.fun} = \{a \mid a : A \text{ and } \text{fun.a} = !\}$

In the sequel we shall also use the notation

$\text{fun}[a/?] = \text{fun truncate-to } (\text{dom.fun} - \{a\})$

Another notation that will be used frequently comes from Haskell Curry and concerns many-argument function whose arguments are taken successively one after another. For instance, if

$\text{fun} : A \rightarrow (B \rightarrow (C \rightarrow (D \rightarrow E)))$  (2.1-2)

then a value of such a function would be traditionally written as

$((((\text{fun.a}).b).c).d)$

but Curry writes it as

$\text{fun.a.b.c.d}$

which intuitively means that

- function  $f$  takes  $a$  as an argument and returns as a value a function  $\text{fun.a}$  that belongs to the set  $B \rightarrow (C \rightarrow (D \rightarrow E))$ , and next
- function  $\text{fun.a}$  takes as an argument an element  $b$  and returns as a function  $\text{fun.a.b}$  that belongs to  $C \rightarrow (D \rightarrow E)$ , etc.

This notation allows not only to avoid many parentheses but also to define function of “mixed” types like e.g.

$\text{fun} : A \rightarrow (B \mapsto (C \rightarrow (D \Rightarrow E)))$     or    (2.1-3)  
 $\text{fun} : (A \rightarrow B) \mapsto (C \rightarrow (D \Rightarrow E))$

Another simplifying convention allows to write

$\text{fun} : A \rightarrow B \mapsto C \rightarrow D \Rightarrow E$  (2.1-4)

instead of

$\text{fun} : A \rightarrow (B \mapsto (C \rightarrow (D \Rightarrow E)))$  (2.1-5)

The expression

$\text{fun} : \mapsto A$  (2.1-6)

means that  $\text{fun}$  is a zero-argument function with only one value that belongs to  $A$ . That value is denoted by

$\text{fun.()}$

About formulas in (2.1-2) to (2.1-6) we say that they describe *type* or *signatures* of corresponding functions. For instance we say that the function in (2.1-4) *is of the type*

$A \rightarrow B \mapsto C \rightarrow D \Rightarrow E$

For every (possibly partial) function

$\text{fun} : A \rightarrow A,$



by its  $n$ -th iteration where  $n = 0, 1, 2, \dots$  we shall mean the function

$$\text{fun}^n : A \rightarrow A$$

defined in the following way:

$\text{fun}^0$  is an *identity function* on  $A$ , i.e.  $\text{fun}^0.a = a$  for every  $a : A$ ,

$\text{fun}^n.a = \text{fun}^n(\text{fun}^{n-1}.a)$  for  $n > 0$ .

In denotational descriptions of programming languages, we shall frequently use many-level *conditional definitions of functions* with the following scheme:

$$\begin{aligned} \text{fun}.x = & \\ & \text{pre-1}.x \rightarrow \text{val-1} \\ & \text{pre-2}.x \rightarrow \text{val-2} \\ & \dots \\ & \text{pre-n}.x \rightarrow \text{val-n} \end{aligned} \tag{2.1-7}$$

where each  $\text{pre-}i$  is a classical predicate, i.e., a total function with logical values  $\text{tt}$  or  $\text{ff}$ , and each  $\text{val-}i$  is some value. Formula (2.1-7) is read as follows:

if  $\text{pre-1}.x$  is true, then  $f.x = \text{val-1}$  and otherwise,

if  $\text{pre-2}.x$  is true, then  $f.x = \text{val-2}$  and otherwise,

...

Intuitively speaking, the evaluation of this function goes line by line and terminates at the first line where  $\text{pre-}i.x$  is satisfied. Of course, to make such a definition unambiguous, the disjunction of all predicates  $\text{pre-}i.x$  must evaluate to “true”, which means that all these predicates must exhaust all cases. Our usual way to ensure this condition will be to write **true** for  $\text{pre-n}.x$  at the last line, which denotes a predicate, that is always true. It can also be read as “in all other cases”.

In the scheme (2.1-7) we also allow the situation where, in the place of a  $\text{val-}i$  we have the undefinedness sign “?” which means that for  $x$  that satisfies  $\text{pre-}i.x$ , function  $f$  is undefined. This convention allows for conditional definitions of partial functions.

In conditional definitions we also use a technique similar to defining local constants in programs. For instance if  $\text{fun} : A \times B \mapsto C$  we can write

$$\begin{aligned} \text{fun}.x = & \\ & \text{pre-1}.x \rightarrow \text{val-1} \\ & \mathbf{let} \\ & \quad (a, b) = x \\ & \text{pre-2}.a \rightarrow \text{val-2} \\ & \text{pre-3}.b \rightarrow \text{val-3} \\ & \dots \end{aligned}$$

which is read as: *let  $x$  be a pair of the form  $(a, b)$* . We can also use **let** in the following way:

$$\begin{aligned} \text{fun}.x = & \\ & \text{pre-1}.x \rightarrow \text{val-1} \\ & \mathbf{let} \\ & \quad y = h.x \\ & \text{pre-2}.x \rightarrow \text{fun-2}.y \\ & \text{pre-3}.x \rightarrow \text{fun-3}.y, \\ & \dots \end{aligned}$$

All these explanations are certainly not very formal, but the notation should be clear when it comes to its applications in concrete cases.

A finite total function  $\text{fun} : \{a_1, \dots, a_n\} \mapsto \{b_1, \dots, b_n\}$  defined by the formula:



```

fun.x =
  x=a-1  → b-1
  x=a-2  → b-2
  ...
  x=a-n  → b-n
  true   → ?

```

shall be written as

$[a-1/b-1, \dots, a-n/b-n]$  or alternatively as  $[a-i/b-i \mid i = 1;n]$ .

The empty function will be denoted by  $[]$ . Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$ . The *overwriting of  $f$  by  $g$*  is a function denoted by

$f \blacklozenge g : A|C \rightarrow B|D$

and defined in the following way:

```

(f ⬢ g).x =
  g.x = !  → g.x
  true    → f.x

```

In particular, if  $f.x=?$  and  $g.x=?$ , then  $f \blacklozenge g.x=?$ . A special case of overwriting is an *update of a function* written as  $f[a-1/b-1, \dots, a-n/b-n]$  and defined by the formula

```

f[a-1/b-1, ..., a-n/b-n].x =
  x = a-1  → b-1
  ...
  x = a-n  → b-n
  true     → f.x

```

We may also overwrite by an undefinedness:

```

(f[a-1/? , ..., a-n/?]).x =
  x = a-1  → ?
  ...
  x = a-n  → ?
  true     → f.x

```

Given two sets of functions  $F$  and  $G$ , we may overwrite one set by the other:

$F \blacklozenge G = \{f \blacklozenge g \mid f : F, g : G\}$ .

## 2.2 Tuples

An expression

$(a-1, \dots, a-n)$  or alternatively  $(a-i \mid i=1;n)$

denotes *n-tuple*. Consequently  $()$  denotes an empty tuple. The difference between tuples and finite sets is such that the order of elements in a tuple is relevant and repetitions are allowed, which is not the case for sets. E.g.

$\{a, b, c, c\} = \{a, c, b\} = \{a, b, c\} = \dots$  but  
 $(a, b, c, c) \neq (a, c, c, b) \neq (a, b, c)$

where  $a, b$  and  $c$  are different with each other.

Tuples are used as mathematical models for several concepts and among others for pushdowns. In this case the following functions will be used later on in the book:

```

push.(b, (a-1, ..., a-n)) = (b, a-1, ..., a-n,)  for n ≥ 0
pop.(a-1, ..., a-n)      = (a-2, ..., a-n)      for n ≥ 2
pop.(a)                  = ()

```

$$\begin{array}{lll}
\text{pop.}() & = () & \\
\text{top.}(a-1, \dots, a-n) & = a-1 & \text{for } n \geq 1 \\
\text{top.}() & = ? & 
\end{array}$$

An important operation on tuples is a *concatenation of tuples*:

$$(a-1, \dots, a-n) \odot (b-1, \dots, b-m) = (a-1, \dots, a-n, b-1, \dots, b-m).$$

We shall also use two predicates:

$$\begin{array}{ll}
\text{are-repetitions.}(a-1, \dots, a-n) & = \text{tt} \quad \text{iff there exist } i \neq j \text{ such that } a-i = a-j \\
\text{no-repetitions.}(a-1, \dots, a-n) & = \text{tt} \quad \text{iff there are no } i \neq j \text{ such that } a-i = a-j
\end{array}$$

Tuples may also be regarded as functions from natural numbers into their elements i.e.

$$(a-1, \dots, a-n).i = a-i$$

In the sequel we shall also need a function that given a tuple, returns its length:

$$\begin{array}{ll}
\text{length.}() & = 0 \\
\text{length.}(a-1, \dots, a-n) & = n
\end{array}$$

and another function that given a tuple returns the set of its element:

$$\text{elements.}(a-1, \dots, a-n) = \{a-1, \dots, a-n\}$$

## 2.3 Partially ordered sets

Let  $A$  be an arbitrary set and let

$$\sqsubseteq : \text{Rel}(A, A)$$

be a binary relation in this set. Relation  $\sqsubseteq$  is said to be a *partial order* in  $A$ , if for any  $a, b, c : A$  the following conditions are satisfied:

1.  $a \sqsubseteq a$  *reflexivity*
2. if  $a \sqsubseteq b$  and  $b \sqsubseteq c$  then  $a \sqsubseteq c$  *transitivity*
3. if  $a \sqsubseteq b$  and  $b \sqsubseteq a$  then  $a = b$  *weak antisymmetry*

If  $a \sqsubseteq b$ , then we say that  $a$  is *smaller* than  $b$  or that  $b$  is *greater* than  $a$ . If additionally  $a \neq b$ , then we say that  $a$  is *significantly smaller* than  $b$  or that  $b$  is *significantly greater* than  $a$ .

A pair  $(A, \sqsubseteq)$  is called a *partially ordered set* (abbr. POS), and the set  $A$  is called its *carrier*. The word “partial” indicates that not necessarily any two elements of  $A$  are comparable with each other. If

for any  $a$  and  $b$  either  $a \sqsubseteq b$  or  $b \sqsubseteq a$ ,

then we say that  $\sqsubseteq$  is a *total order*.

Of course, every total order is partial but not vice versa. An example of a partial order which is not total is the inclusion of sets. Such POS is called *set-theoretic POS*.

Let  $B$  be a subset of a partially ordered set  $A$  and let  $b : B$ . In this case

- $b$  is called a *minimal element* in  $B$ , if there is no  $a : B$  such that  $a \sqsubseteq b$  and  $a \neq b$
- $b$  is called the *least element* in  $B$ , if for any  $a : B$  holds  $b \sqsubseteq a$ ,
- $b$  is called a *maximal element* in  $B$ , if there is no  $a : B$  such that  $b \sqsubseteq a$  and  $a \neq b$ ,
- $b$  is called the *greatest element* in  $B$ , if for any  $a : B$  holds  $a \sqsubseteq b$ .

There exist partially ordered sets without a minimal element and sets where there is more than one such element. However, if there is the least element in a set, then it is the unique minimal element and analogously for maximal and greatest elements.

An *upper bound* of  $B$  is such an element of  $A$ , which is greater than any element of  $B$ . Notice that an upper bound of a set does not need to belong to that set, but if it does, then it is the greatest element of the set.

If the set of all upper bounds of  $B$  has the least element, then this element is called the *least upper bound* of  $B$ <sup>1</sup>. If a two-element set  $\{a, b\}$  has the least upper bound, then we denote it by

$$a \vee b$$

In a set-theoretic POS, the least upper bound of a family of sets is the set-theoretic union of that family. This, of course, also concerns a family of two sets.

## 2.4 Chain-complete partially-ordered sets

Let  $(A, \sqsubseteq)$  be a partially ordered set. By a *chain* in that set we mean any sequence of elements of  $A$ :

$$a.1, a.2, a.3, \dots$$

such that  $a.i \sqsubseteq a.(i+1)$ . Here, for a change we write  $a.i$  instead of  $a-i$ . Note that in this case  $a$  may be regarded as a function with natural-number arguments. If the set of all elements of a chain has the least upper bound, then it is called the *limit* of that chain and is denoted by:

$$\lim.(a.i \mid i = 1, 2, \dots)$$

A POS is said to be *chain-complete partially ordered set* (abbr. CPO) if:

1. every chain in  $A$  has a limit,
2. there exists the least element in  $A$ .

This least element we shall denote by  $\Phi$ .

A total function  $f : A \mapsto A$  is said to be *monotone* if  $a \sqsubseteq b$  implies  $f.a \sqsubseteq f.b$  and we say that it is *continuous* if the following two conditions are satisfied:

1. for any chain  $(a.i \mid i = 1, 2, \dots)$  the sequence  $(f.(a.i) \mid i = 1, 2, \dots)$  is also a chain,
2. if the former has a limit, then the latter has a limit as well and

$$\lim.(f.(a.i) \mid i = 1, 2, \dots) = f.(\lim.(a.i \mid i = 1, 2, \dots)).$$

As is easy to see, every continuous function is monotone, which follows from the fact that

$$\text{if } a \sqsubseteq b \text{ then } \lim(a, b, b, b, \dots) = b.$$

Continuous functions satisfy a so called *Kleene theorem* (see [64]) — which we shall frequently use in our applications.

**Theorem 2.4-1** *If  $f$  is continuous in a chain-complete set, then the set of all solutions of the equation*

$$x = f.x \tag{2.4-1}$$

*is not empty and contains the least element defined by the equation*

$$Y.f = \lim.(f^n.\Phi \mid n = 0, 1, 2, \dots) \blacksquare$$

**Proof** of that theorem is very simple:

$$f.(Y.f) = f.(\lim.(f^n.\Phi \mid n = 0, 1, 2, \dots)) = \lim.(f^{n+1}.\Phi \mid n = 0, 1, 2, \dots) = \lim.(f^n.\Phi \mid n = 0, 1, 2, \dots).$$

The last equality follows from the fact that  $f^0.\Phi = \Phi$ , hence adding  $f^0.\Phi$  to the chain, does not change its limit. The property that  $Y.f$  is the least fixed point follows from the fact that for any other fixpoint  $X.f$ ,  $\Phi \sqsubseteq X.f$  and from the monotonicity of  $f$  we have  $f^n.\Phi \sqsubseteq X.f$  hence  $\lim.(f^n.\Phi \mid n = 0, 1, 2, \dots) \sqsubseteq X.f$ . ■

The equation (2.4-1) is called a *fixed point equation* and its solution  $Y.f$  — the *least fixed point of function  $f$* . It is the least solution of the equation (2.4-1), but in the sequel we will call it simply *the solution* since other solutions will not be concerned.

---

<sup>1</sup> The greatest lower bound is defined in an analogous way but we will not need this concept in the book.

The concept of a one-argument continuous function may be simply generalized to functions of many arguments. We say that

$$f : A^{cn} \mapsto A \quad (2.4-2)$$

is *continuous wrt to its first element*, if for any tuple  $(a.1, \dots, a.(n-1))$  the function

$$g.a = f.(a, a.1, \dots, a.(n-1))$$

is continuous. In an analogous way we define the continuity of  $f$  with respect to any other of its arguments.

A many-argument function (2.4-2) is called *continuous* if it is continuous in all of its arguments.

As we are going to see soon, continuous functions are fundamental for our applications since due to Kleene's theorem we can recursively define sets and functions. Such definitions will most frequently have the form

$$x.1 = f.1.(x.1, \dots, x.n)$$

...

$$x.n = f.n.(x.1, \dots, x.n)$$

Of course, every such set of equations may be regarded as one equation

$$X = f.X$$

in a POS over a Cartesian product  $A.1 \times \dots \times A.n$  where

$$f.(x.1, \dots, x.n) = (f.1.(x.1, \dots, x.n), \dots, f.n.(x.1, \dots, x.n))$$

and where the order is defined component-wise, i.e.

$$(a.1, \dots, a.n) \sqsubseteq (b.1, \dots, b.n) \text{ iff } a.i \sqsubseteq b.i \text{ for } i = 1;n.$$

As is easy to show, if all  $A.i$ 's are chain-complete, then their Cartesian product is chain-complete wrt the above order. Besides, if all  $f.i$  are continuous, then  $f$  is continuous, as well.

As turns out, fixed-point sets of equations with continuous functions may be transformed (and reduced) in a way analogous to the case of algebraic equations. It is expressed by two theorems due to Hans Bekić [12] and Jacek Leszczyłowski [67].

**Theorem 2.4-2** *If  $f, g : A \times A \mapsto A$  are continuous, then the set of equations*

$$a = f.(a, b)$$

$$b = g.(a, b)$$

*is equivalent to*

$$a = f.(a, b)$$

$$b = g.(f.(a, b), b) \quad \blacksquare$$

**Theorem 2.4-3** *If  $f, g : A \times A \mapsto A$  are continuous, then the set of equations*

$$a = f.(a, b)$$

$$b = g.(a, b)$$

*is equivalent to*

$$a = h.b$$

$$b = g.(a, b)$$

*where  $h$  is a function that to every  $b$  assigns the least fixed point of  $f.(x, b)$  regarded as a one-argument function of  $x$  running over the set  $A$ . ■*

As we are going to see, the theory of fixed-point equations in CPO is an important tool for writing recursive definitions of sets and of functions in denotational models.

## 2.5 A CPO of formal languages

Grammars of *natural languages* such as English, Polish or French may be regarded as algorithms allowing to check which sentences are grammatically correct and which are not. In this spirit Noam Chomsky<sup>2</sup> has developed in early 1960. his model of *generative context-free grammars* or simply *context-free grammars* (see [43] – [47]). Formal languages generable by such grammars have been called *context-free languages*.

Although his model turned out to be not write adequate for natural languages, it was successfully applied to programming languages. In the early years for Algol 60 and Pascal, later for ADA and CHILL and many other languages. These applications contributed to a rapid development of their theory. The first internationally recognized monography on that subject was written in 1966 by Seymour Ginsburg [55], and the first Polish monography in 1971 by Andrzej Blikle [17]. A year later, Andrzej Blikle has published a paper on *equational grammars* [19], which are equivalent, in a sense, to context-free grammars.

This section contains a short introduction to context-free languages in the context of equational grammars which are discussed in Sec. 2.6.

Let  $A$  be an arbitrary finite set of symbols called an *alphabet*. By a *word* over  $A$ , we mean every finite tuple over  $A$ , including the empty tuple  $()$ . Traditionally words are written as sequences of characters, e.g.,  $accbda$ .

Since words are tuples (of characters) we can apply to them the operation of concatenation defined in Sec. 2.2. E.g.

$$abdaa \odot eaag = abdaaeaag$$

Every set  $L$  of words over  $A$  is called a *formal language* (or simply a *language*) over  $A$ . By  $\text{Lan}.A$  we denote the family of all languages over  $A$  and by  $\{\}$  — an empty language (empty set). If  $P$  and  $Q$  are languages, then their *concatenation* is the language defined by the equation:

$$P \odot Q = \{p \odot q \mid p:P \text{ and } q:Q\}.$$

As we see, by  $\odot$  we denote not only a function on words but also on languages. If it does not lead to ambiguities,  $P \odot Q$  is written as  $PQ$ . Since concatenation is an associative operation, we can write  $PQL$  instead of  $(PQ)L$  or  $P(QL)$ . We shall also assume that concatenation binds stronger than set-theoretic union, hence instead of

$$(P \odot Q) \mid (R \odot S)$$

we shall write

$$PQ \mid RS.$$

It is also easy to see that concatenation is left- and right-distributive over the union, i.e.

$$\begin{aligned} (P \mid Q) R &= PR \mid QR \\ R (P \mid Q) &= RP \mid RQ \end{aligned}$$

The  $n$ -th *power* of a language  $P$  is defined recursively:

$$\begin{aligned} P^0 &= \{ () \} \\ P^n &= P \odot P^{n-1} \text{ for } n > 0 \end{aligned}$$

We shall also use two operators called respectively (language-theoretic) *plus* and *star*:

$$\begin{aligned} P^+ &= U.\{P^i \mid i > 0\} \\ P^* &= P^+ \mid P^0 \end{aligned}$$

Hence for an alphabet  $A$ , the set  $A^+$  is the set of all non-empty words over  $A$ , and  $A^*$  is the set of all words over  $A$ . Languages over  $A$  are subsets of  $A^*$ .

---

<sup>2</sup> Noam Chomsky — an American linguist, philosopher and political activist. Professor of linguistics at Massachusetts Institute of Technology, creator of the concept of generative grammars.

Note the difference between  $L^*$  and  $L^c$ . Whereas the former is a set of words over the alphabet of  $L$ , the latter is a set of tuples of words of  $L$ .

Since the inclusion of sets is a partial order,  $(\text{Lan}.A, \subseteq)$  is a CPO with empty language as the least element. As is easy to show, all operations on languages, which are defined above, plus the union of languages, are continuous. For any two languages,  $P$  and  $Q$ , their least upper bound is their union  $P \mid Q$ , and the limit of a chain of languages is the union of the elements of the chain.

## 2.6 Equational grammars

Since all the operations on languages defined in Sec. 2.5 are continuous, they can be used in fixed-point equations (Sec. 2.4) regarded as grammars. This idea is elaborated below.

Consider a simple example of a set of equations that defines the set of identifiers of a programming language. In our example we assume that identifiers always start from a letter:

```

Letter    = {a, b, ..., z}
Digit     = {0, 1, ..., 9}
Character = Letter | Digit
Suffix    = {()} | Character © Suffix
Identifier = Letter © Suffix

```

Such sets of equations are called *equational grammars*, and their solutions (tuples of languages) are called *many-sorted languages*. In the above case, the defined many-sorted language is a tuple of five categories (sorts):

(Letter, Digit, Character, Suffix, Identifier).

The category **Suffix** has an auxiliary character since its only role is to express the fact that an identifier must start with a letter. Its equation can be eliminated in using the Theorem 2.3-2 and the Theorem 2.3-3. As is easy to prove

$\text{Suffix} = \text{Character}^*$

hence our grammar may be reduced to a more compact form

```

Letter    = {a, b, ..., z}
Digit     = {0, 1, ..., 9}
Identifier = Letter © (Letter | Digit)*

```

This grammar defines a many-sorted language, which consists of three categories — and therefore is different from the former — but defines the same set **Identifier**.

Let us now investigate equational grammars more formally (for details see [19]). Let  $A$  be an arbitrary non-empty finite alphabet and let

$\text{Fam} \subseteq \text{Lan}.A$

be an arbitrary family of languages over  $A$ . Let  $\text{Pol.Fam}$  denotes the least class of functions of the type:

$p : (\text{Lan}.A)^{cn} \mapsto \text{Lan}.A \quad \text{where } n \geq 0$

which contains:

- (1) all projections, i.e. functions of the form  $f.(X.1, \dots, X.n) = X.i$  for  $i \leq n$ ,
- (2) all functions with constant values in  $\text{Fam}$ ,
- (3) the union and concatenation of languages

and is closed over the composition (superposition) of functions.

Functions in  $\text{Pol.Fam}$  are called *polynomials over Fam*. Since all functions described in (1), (2) and (3) are continuous, and a composition of continuous functions is continuous, all polynomials are continuous.

By an *atomic language* over  $A$  we shall mean any one-element language  $\{w\}$ , where  $w : A^*$ . Polynomials over an arbitrary set of atomic languages are called *Chomsky's polynomials*. Below a few examples of such polynomials:

$$\begin{aligned} p_1.(X,Y,Z) &= \{b\} \\ p_2.(X,Y) &= \{b\} \\ p_3.(X,Y,Z) &= X \\ p_4.(X,Y,Z) &= (\{d\}X\{b\}YY\{c\} \mid X) Z \end{aligned}$$

Observe that for a complete identification of a polynomial we have to define its arity. This can be seen on the examples of  $p_1$  and  $p_2$  which are different although return the same language.

Polynomials which do not “contain” union — e.g., such as  $p_1$ ,  $p_2$ , and  $p_3$  — are called *monomials*. Since concatenation is distributive over union, every polynomial may be reduced to a union of monomials.

An *equational grammar* over an alphabet  $A$  is any fixed-point set of equations of the form:

$$\begin{aligned} X.1 &= p_1.(X.1, \dots, X.n) \\ &\dots \\ X.n &= p_n.(X.1, \dots, X.n) \end{aligned}$$

where all  $p_i$ 's are Chomsky's polynomials over  $A$ . Since polynomials are continuous, this set of equations has a unique least solution  $(L.1, \dots, L.n)$ . The languages  $L.1, \dots, L.n$  are said to be *defined by our grammar*. We also say that they are *equationally definable*.

As has been proved in [19], the class of equationally-definable languages is identical with the class of *context-free languages* in the sense of Chomsky<sup>3</sup>. Such a class remains the same if we allow the operations “\*” and “+” in polynomials and if polynomials are built over arbitrary equationally-definable languages. For proofs of all these facts, see [19].

Due to these facts in the sequel, equationally-definable languages will be called *context-free*.

## 2.7 A CPO of binary relations

Let  $A$  and  $B$  be arbitrary sets. Any subset of their Cartesian product  $A \times B$  will be called a *binary relation* or just a *relation* between these sets. Hence

$$\text{Rel}(A,B) = \{R \mid R \subseteq A \times B\}$$

is the set of all binary relations between  $A$  and  $B$ . Instead of writing  $(a,b) : R$ , we shall usually write  $a R b$ .

If  $A = B$ , then instead of  $\text{Rel}(A, A)$  we write  $\text{Rel}(A)$ . For every  $A$  we define an *identity relation*:

$$[A] = \{(a, a) \mid a:A\}$$

By  $\emptyset$ , we shall denote the *empty relation*. Let now

$$\begin{aligned} \text{Boolean} &= \{tt, ff\} && \text{— logical values} \\ p : A &\rightarrow \text{Boolean} && \text{— a predicate} \end{aligned}$$

With every predicate, we assign an identity relation defined by

$$\text{Id}(p) = \{(a, a) \mid p.a = tt\}$$

If  $R : \text{Rel}(A,B)$ , then

$$\begin{aligned} \text{dom}.R &= \{a \mid (\exists b : B) a R b\} && \text{— the domain of } R \\ \text{cod}.R &= \{b \mid (\exists a : A) a R b\} && \text{— the codomain of } R \end{aligned}$$

Let  $P : \text{Rel}(A,B)$  and  $R : \text{Rel}(B,C)$ . A *sequential composition* of  $P$  and  $R$  is a relation

<sup>3</sup> Which means that for each equational grammar there exists an equivalent context-free grammar and vice versa.



$$P \bullet R : \text{Rel}(A, C)$$

defined as follows:

$$P \bullet R = \{(a, c) \mid (\exists b : B) (a P b \ \& \ b R c)\}$$

For every two relations, their composition always exists, although it may be an empty relation. As is easy to check  $\bullet$  is associative i.e.

$$(P \bullet R) \bullet Q = P \bullet (R \bullet Q)$$

It is, therefore, legal to write  $P \bullet R \bullet Q$ . We shall also write  $PR$  instead of  $P \bullet R$  whenever this does not lead to misunderstanding, and we shall assume that composition binds stronger than union, hence instead of

$$(P \bullet R) \mid (Q \bullet S)$$

we write

$$PR \mid QS.$$

In the sequel, the sequential composition of relations will be frequently applied in the case where the composed relations are function. In that case:

$$(P \bullet R).a = R.(P.a)$$

and therefore

$$(P \bullet R \bullet Q).a = (P \bullet (R \bullet Q)).a = Q.(R.(P.a))$$

which means that in a sequential composition of functions, the composed functions are “executed” from left to right one after another.

Similarly as for languages, also for relations, we define the operations of power, plus and star:

$$\begin{aligned} R^0 &= [A] && \text{identity relation in over } A \\ R^n &= RR^{n-1} \text{ for } n > 0 \\ R^+ &= \bigcup \{R^n \mid n > 0\} \\ R^* &= R^+ \mid R^0 \end{aligned}$$

The *converse relation* for  $R$  is defined as follows

$$a R^{-1} b \quad \text{iff} \quad b R a$$

A relation  $R$  is called a *function*, if

for any  $a, b$  and  $c$ , if  $a R b$  and  $a R c$ , then  $b = c$ .

If  $R$  and  $R^{-1}$  are functions, then  $R$  is said to be a *convertible function* or a *one-one function*. If  $P$  and  $R$  are functions, then  $PR$  is also a function and

$$(PR).a = P.(R.a)$$

hence the composition of functions is their superposition.

The set of relations  $\text{Rel}(A, B)$  constitutes a CPO with ordering by set-theoretic inclusion and the empty relation as the least element. All of the defined operations on relations are continuous. In the sequel we shall frequently refer to the following theorem:

**Theorem 2.7-1** For any  $P, Q : \text{Rel}(A)$  the least solutions of equations

$$\begin{aligned} X &= P \mid QX && \text{and} \\ X &= P \mid XQ \end{aligned}$$

are respectively

$$X = Q^*P \quad \text{and}$$



$$X = P * Q$$

Moreover, if both  $P$  and  $R$  are functions with disjoint domains, then both these solutions are also functions. ■

In this place, it is worth noticing that the set of partial functions

$$A \rightarrow B$$

constitutes a chain-complete subset of  $(\text{Rel}(A, B), \subseteq)$  that is closed under the composition of arbitrary functions and union of functions with disjoint domains. Of course, both these operations are continuous.

Due to these facts, functions can be defined by fixed-point (recursive) equations. Since  $A$  and  $B$  are arbitrary, this is also true for functions of type

$$f : A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n$$

provided that appropriate constructors are defined. As a first example, consider a recursive definition of a function of an  $n$ -th power of number 2, i.e.<sup>4</sup>.

$$\begin{aligned} \text{power-of-two} : \text{Number} \rightarrow \text{Number} & \quad \text{where } \text{Number} = \{0, 1, 2, \dots\} \\ \text{power-of-two}.n = 2^n & \quad \text{for an integer } n \geq 0 \end{aligned}$$

A recursive definition of that function is as follows:

$$\begin{aligned} \text{power-of-two}.n = \\ n = 0 & \rightarrow 1 \\ n > 0 & \rightarrow \text{power-of-two}.(n-1) * 2 \end{aligned}$$

This definition written as a fixed-point equation in the set-theoretic CPO

$$(\text{Number} \rightarrow \text{Number}, \subseteq, [ \ ])$$

is as follows

$$\text{power-of-two} = \text{zero} \blacklozenge (\text{minus} \bullet \text{power-of-two}) \bullet \text{double}$$

where

$$\begin{aligned} \text{zero}.n &= [0/1] \\ \text{minus}.n &= n-1 \quad \text{for } n > 0 \\ \text{minus}.0 &= ? \\ \text{double}.n &= 2 * n \end{aligned}$$

Notice that all these functions are constants in our equation, hence the right-hand side of that equation represents a one-argument function in our CPO:

$$F.\text{fun} = \text{zero} \blacklozenge (\text{minus} \bullet \text{fun}) \bullet \text{double}$$

Since, as is easy to prove,  $\blacklozenge$  and  $\bullet$  are continuous on both arguments, our function  $F$  is continuous as well, and therefore — according to Kleene's theorem — the least solution of our equation is the limit (the union) of the following chain of functions:

$$\begin{aligned} F.\{ \} &= \text{zero} & &= [0/1] \\ F.\text{zero} &= \text{zero} \blacklozenge (\text{minus} \bullet \text{zero}) \bullet \text{double} &= [0/1, 1/2] \\ F.(F.\text{zero}) &= \text{zero} \blacklozenge (\text{minus} \bullet F.\text{zero}) \bullet \text{double} &= [0/1, 1/2, 2/4] \\ &\dots \end{aligned}$$

Each element of that chain is a finite approximation of our function **power-of-two**.

Now let us consider a technically more complicated example of a two-argument function of power in the set of natural numbers:

---

<sup>4</sup> Here we introduce a notational convention of VDM and MetaSoft where instead of using one-character symbols as in usual mathematics, we use many-character symbols for both sets and functions. As we are going to see later, this convention is practically a must in the case of denotational models where numbers of symbols goes into tens if not hundreds.

$\text{power} : \text{Number} \times \text{Number} \rightarrow \text{Number}$

$\text{power}.n.m =$

$m = 0 \rightarrow 1$

$m > 0 \rightarrow n * \text{power}.n.(m-1)$

Also this definition can be expressed as a fixed-point equation in the CPO of binary relations:

$\text{Rel.}(\text{Number} \times \text{Number}, \text{Number})$

To see that, let us construct a fixed-point equation whose solution is the function:

$\text{power}.(n, m) = n^m$

regarded as a relation in our CPO. Let us start from the definitions of a certain operation of composition of functions

$F, Q : \text{Rel.}(A \times A, A).$  (2.7-1)

By the *composition of F and Q on the second argument*, we shall mean the relation

$F \circledast Q = \{((a, b), c) \mid (\exists d) ((a, b), d) : F \text{ and } ((a, d), c) : Q\}$

If F and Q are functions then

$[F \circledast Q].(a, b) = Q.(a, F.(a, b))$

The set of relations (2.7-1) is, of course, a CPO with set-theoretic inclusion. One can show that  $\circledast$  is continuous on both arguments. Since the limit of a chain is in our case the set-theoretic union, it is sufficient to show that  $\circledast$  is distributive over union on both arguments, which means that the following equalities hold (we assume that  $\circledast$  binds stronger than the union):

$(F-1 \mid F-2) \circledast Q = F-1 \circledast Q \mid F-2 \circledast Q$  and

$F \circledast (Q-1 \mid Q-2) = F \circledast Q-1 \mid F \circledast Q-2$

Let then

$((a, b), c) : (F-1 \mid F-2) \circledast Q$

which means that there exists a d such that

$((a, b), d) : (F-1 \mid F-2) \text{ and } ((a, d), c) : Q$

which means that there exist i and d such that

$((a, b), d) : F-i \text{ and } ((a, d), c) : Q$

which means that there exists i such that

$((a, b), c) : F-i \circledast Q$

which means that

$((a, b), c) : F-1 \circledast Q \mid F-2 \circledast Q$

In this way, we have proved the inclusion

$(F-1 \mid F-2) \circledast Q \subseteq F \circledast Q-1 \mid F \circledast Q-2$

The proofs of the remaining three inclusions are analogous.

Since  $\circledast$  is continuous on both arguments the following fixed-point equation has the least solution:

$\text{power} = \text{zero} \mid (\text{minus} \circledast \text{power}) \circledast \text{times}$  (2.7-2)

where:

$\text{zero}(n, 0) = 1$

$\text{minus}.(n, m) = m-1$  for  $m > 0$ , and for  $m = 0$  this function is undefined

$$\text{times.}(n, m) = n * m$$

Since the set-theoretic union and our composition are both continuous in the CPO of relations (2.7-1), Kleene's theorem implies that the solution of (2.7-2) is the limit of the chain of relation

$$P^0 \subseteq P^1 \subseteq P^2 \subseteq \dots \quad (2.7-3)$$

which are functions defined in the following way:

$$\begin{aligned} P^0 &= \text{zero} \\ P^{i+1} &= (\text{minus } \otimes P^i) \otimes \text{times} \quad \text{for } i \geq 0 \end{aligned}$$

This means that for every  $i \geq 0$  function  $P^i$  is a partial function of power restricted to  $m \leq i$ :

$$\begin{aligned} P^i.(n, m) &= \\ m \leq i &\rightarrow m^i \\ \text{true} &\rightarrow ? \end{aligned}$$

Since all these functions coincide on the common parts of their domains, the set-theoretic union of the chain (2.7-3) is a function, and it is the power function.

## 2.8 A CPO of denotational domains

One of the main tools of denotational models of software systems are sets traditionally called *domains*. These domains are most frequently defined using equations — sometimes fixed-point equations — based on functions that are listed below. Some of them have been already defined, but we recall their descriptions to have their full list in one place:

1.  $A \mid B$  — set-theoretic union
2.  $A \cap B$  — set-theoretic intersection
3.  $A \times B$  — Cartesian product
4.  $A^{c^n}$  — Cartesian  $n$ -th power
5.  $A^{c^+}$  — Cartesian  $+$ -iteration
6.  $A^{c^*}$  — Cartesian  $*$ -iteration
7.  $\text{FinSub}.A$  — the set of all finite subsets
8.  $A \Rightarrow B$  — the set of all mappings including the empty mapping
9.  $A - B$  — set-theoretic difference
10.  $\text{Sub}.A$  — the set of all subsets
11.  $A \rightarrow B$  — the set of all functions from  $A$  to  $B$
12.  $A \mapsto B$  — the set of all total functions from  $A$  to  $B$
13.  $\text{Rel.}(A, B)$  — the set of all relations between  $A$  and  $B$

These operators may be used in not-recursive equations, e.g.:

$$\begin{aligned} \text{State} &= \text{Identifier} \Rightarrow \text{Data} \\ \text{InsDen} &= \text{State} \rightarrow \text{State} \end{aligned} \quad \begin{array}{l} (2.8-1) \\ \text{instruction denotations} \end{array}$$

where  $\text{InsDen}$  denotes a domain of the denotations of instructions, or in fixed point equations, e.g.:

$$\begin{aligned} \text{Record} &= \text{Identifier} \Rightarrow \text{Data} \\ \text{Data} &= \text{Number} \mid \text{Record}. \end{aligned} \quad (2.8-2)$$

Whereas definition (2.8-1) does not raises any doubts, in the case of (2.8-2) the situation is different. Since it is obviously a fixed-point equation we have to prove the continuity of  $\Rightarrow$  and  $\mid$ , but the continuity where? What is the CPO of domains? Set-theoretic inclusion is clearly its partial order, but what is the carrier?

Potentially that carrier should include all domains that we shall define in the future, hence something like the set of all sets. Unfortunately — as has been known since 1899<sup>5</sup> — such a set does not exist<sup>6</sup>. Despite this fact, our problem can be solved on the base of M.P. Cohn’s [48] construction. As he has shown, for any set of sets **B** there exists a set of sets **Set.B** with the following properties:

1. all sets in **B** belong (as elements) to **Set.B**,
2. **Set.B** is closed under all our operations from 1) to 13),
3. **Set.B** is closed under unions of all denumerable families of its elements,
4. the empty set  $\{\}$  belongs to **Set.B**.

Following this construction, we choose as family **B**, the set of all “initial” domains that we shall use in our model, such as **Boolean**, **Number**, **Identifier**, **Character**, etc. Since  $(\text{Set.B}, \sqsubseteq)$  is a set-theoretic CPO, we can talk about the continuity of functions defined on sets in **Set.B**. As is easy to show operations from 1) to 8) are continuous, the difference of sets is continuous only on the left argument, and the remaining functions are not continuous, and therefore they cannot appear in fixed-point equations<sup>7</sup>.

On this ground we can claim that the equation (2.8-2) a least solution)= defined by the theorem of Kleene (Sec.2.4). Records defined in that way may “carry” other records, but of a “lower-level” than themselves. At the end of that hierarchy, we have records carrying numbers. If however, we replace  $\Rightarrow$  by  $\rightarrow$ , then we can’t apply Kleene’s theorem to it. More on that subject in Sec. 3.1.

The fact that non-continuous operators can’t be used in a fixed-point context does not mean that they cannot be used in fixed-point equations “at all”. For instance, our two sets of equations (2.8-1) and (2.8-2) can be legally combined into one:

$$\begin{array}{llll} \text{dat} : \text{Data} & = \text{Number} & | & \text{Record} \\ \text{rec} : \text{Record} & = \text{Identifier} & \Rightarrow & \text{Data} \\ \text{sta} : \text{State} & = \text{Identifier} & \Rightarrow & \text{Data} \\ \text{ind} : \text{InsDen} & = \text{State} & \rightarrow & \text{State} \end{array} \quad (2.8-3)$$

Although “as a whole” this is a fixed-point set of equations with one non-continuous operation, the recursion is present only in the second and the third equation where the operators are continuous. This set of equations is therefore “legal”, and the existence of its least solution is guaranteed by Kleene’s theorem.

## 2.9 Abstract errors

For practically all expressions appearing in programs, their values in some circumstances can’t be computed “successfully”. Here are a few examples:

- expression  $x/y$  cannot be evaluated if the variables  $x$  or  $y$  have not been declared,

<sup>5</sup> The concept that a set of all sets does not exist is tied to Russell’s paradox, which was published by the British philosopher and mathematician Bertrand Russell in 1911. However, the paradox had already been discovered independently in 1899 by the German mathematician Ernst Zermelo. At the end of the 1890s, Georg Cantor, considered the founder of modern set theory, had already realized that his theory would lead to a contradiction. (credit to Bing)

<sup>6</sup> Formally speaking the attempt of constructing such a set leads to a contradiction. Indeed, let  $Z$  be the set of all sets. Let then  $Z_e$  be the set of all sets that are their own elements and  $Z_n$  — the set of all sets that are not their own elements. Since obviously  $Z = Z_e \cup Z_n$ , set  $Z_n$  must belong to either  $Z_e$  or  $Z_n$ . The first case must be excluded since in that case  $Z_n$  should belong to  $Z_n$ . The second case is impossible either, since then  $Z_n$  must not belong to itself. Intuitively speaking one can say that the collection of all sets is “too large to be a set”.

<sup>7</sup> As an example, let us show that the operator  $\rightarrow$  is not continuous. Let then  $A_1 \subset A_2 \subset \dots$  be an arbitrary chain of mutually different sets, and let  $B$  be an arbitrary set. The sequence of domains  $A_i \rightarrow B$  constitutes a chain but none of its elements contain a total function on the union  $\bigcup A_i$ , hence none of such functions belong to  $U(A_i \rightarrow B)$ , which means that  $U(A_i \rightarrow B) \neq U(\bigcup A_i \rightarrow B)$ . In an analogous way we may show the non-continuity of the operators  $A \mapsto B$  and  $\text{Rel.}(A, B)$ . Notice, however, that  $U(A_i \Rightarrow B) = U(\bigcup A_i \Rightarrow B)$ , and similarly for the right-hand-side argument which means that  $\Rightarrow$  is continuous on both arguments.

The decision of not using non-continuous functions in fixed-point equations is due to the fact that Kleene’s theorem is not satisfied in such cases, and, therefore, fixed-point equations do not correspond to recursion.

- expression  $x/y$  cannot be evaluated if the variables  $x$  or  $y$  have not been declared as numbers,
- expression  $x/y$  cannot be evaluated if the current value of  $y$  is zero,
- expression  $x+y$  cannot be evaluated if its value exceeds the maximal number allowed in current implementation; alternatively, if additions is a modulo operation, it will return an incorrect result,
- the value of the array expression  $a[k]$  cannot be computed if the variable  $a$  has not been declared as an array or if  $k$  is out of the domain of  $a$ ,
- the query “Has John Smith retired?” cannot be answered if John Smith is not listed in the database.

In all these cases, a well-designed implementation should stop the execution of a program and generate an error message.

To describe such a mechanism formally, we introduce the concept of an *abstract error*. In a general case, abstract errors may be anything, but in our models, they will be words, such as, e.g.

‘division by zero not allowed’.

They are closed in apostrophes to distinguish them from metavariables at the level of **MetaSoft**.

The fact that an attempt to evaluate the expression  $x/0$  raises an error message can be now expressed by the equation:

$$x/0 = \text{'division by zero not allowed'}$$

In the general case with every domain **Data**, we shall associate a corresponding domain with abstract errors

$$\text{DataE} = \text{Data} \mid \text{Error}$$

where **Error** is a universal set of all abstract errors that may be generated in course of the executions of our programs. This set will be regarded as a parameter of our denotational model. Now, every partial operation

$$\text{op} : \text{Data}.1 \times \dots \times \text{Data}.n \rightarrow \text{Data},$$

whose partiality is *computable*,<sup>8</sup> may be extended to a total operation

$$\text{ope} : \text{DataE}.1 \times \dots \times \text{DataE}.n \mapsto \text{DataE}$$

Of course  $\text{ope}$  should coincide with  $\text{op}$  wherever  $\text{op}$  is defined, i.e. if  $d.1, \dots, d.n$  are not errors and  $\text{op}(d.1, \dots, d.n)$  is defined, then  $\text{ope}(d.1, \dots, d.n) = \text{op}(d.1, \dots, d.n)$ .

An operation  $\text{ope}$  will be said to be *transparent for errors* or simply *transparent* if the following condition is satisfied:

$$\text{if } d.k \text{ is the first error in the sequence } d.1, \dots, d.n, \text{ then } \text{ope}(d.1, \dots, d.n) = d.k$$

This rule indicates that the arguments of  $\text{ope}$  are evaluated one-by-one from left to right, and the first error (if it appears) becomes the final value of the computation.

The majority of operations on data that will appear in our models will be transparent. An exception are boolean operations discussed in Sec. 2.10.

Error-handling mechanisms are frequently implemented in such a way that errors serve only to inform the user that (and why) program evaluation has been aborted. Such a mechanism will be called *reactive*. However, in some applications the generation of an error may initiate a recovery action. Such mechanisms will be called *proactive*.

---

<sup>8</sup> Partiality of a function  $f$  is computable, if there is an algorithm which for every element  $x$  can detect if  $f.x$  is defined or not. In the examples of this section all functions have computable partialities. However, it is a well-known fact, that in the general case the definedness of recursive functions is not computable. E.g. there is no algorithm which given a program, and a memory state, will check whether the execution of this program starting from this state will terminate. Consequently, we can't assume that any undefinedness will be signaled by an error message.

As we shall see in the sequel, a reactive mechanism may be quite simply enriched to a proactive one. However, since the latter is technically more complicated, in the development of our example-language **Lingua**, except **Lingua-SQL**, we shall most frequently apply a reactive model. Proactive constructions are discussed in Sec. 6.5.3 and Sec. 11.3.4.3.

A well-defined error-handling mechanism allows avoiding situations where programs hang up without any explanation, or even worse — when they generate an incorrect result without warning the user (see Sec. 10.2).

## 2.10 Two three-valued propositional calculi

Tertium non datur — used to say ancients masters. Computers denied this principle.

In the Aristotelean logic, every sentence is either true or false. The third possibility does not exist. However, in the world of computers the third possibility is not only possible but inevitable. For instance the boolean expression  $x/y > 2$  may evaluate to **true**, **false** or error if  $y = 0$ . Error is, therefore, the third logical value.

To describe the error-handling mechanism in boolean expressions the basic domain of two boolean values “true” and “false”:

$\text{Boolean} = \{\text{tt}, \text{ff}\}$

must be enriched by a third element

$\text{BooleanE} = \{\text{tt}, \text{ff}, \text{ee}\}$

where **ee** stands for “error” or an undefinedness caused by an infinite execution. Infinite executions in boolean expressions may happen these expressions may include calls of functional procedures, which may loop.

We assume for simplicity that there is only one error element, since at the level of boolean expressions, all errors will be treated in the same way<sup>9</sup>. How are we handling non-computable undefinednesses will be seen a little later.

Let’s observe now that the transparency of boolean operators would not be an adequate choice. To see that consider a conditional instruction:

**if  $x \neq 0$  and  $1/x < 10$  then  $x := x+1$  else  $x := x-1$  fi**

We would probably expect that for  $x=0$ , one should execute the assignment  $x:=x-1$ . If however, our conjunction would be transparent, then the expression

**$x \neq 0$  and  $1/x < 10$**

would evaluate to ‘division by zero not allowed’, which means that our program would abort. Notice also that the transparency of **and** would imply

**ff and ee = ee**

which would mean that when an interpreter evaluates **p and q**, then it first evaluates both **p** and **q** — as in “usual mathematics” — and only later applies **and** to them. Such a mode is called an *eager evaluation*.

An alternative to it is a *lazy evaluation* where, if **p = ff**, then the evaluation of **q** is abandoned, and the final value of the expression is **ff**. In such a case:

**ff and ee = ff**

**tt or ee = tt**

A three-valued propositional calculus with lazy evaluation was described in 1961 by John McCarthy [78], who defined boolean operators as in Tab. 2.10-1.

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<sup>9</sup> Precisely speaking that is the case of a *reactive error elaboration*, i.e. all errors, except undefinedness, are displayed and abort program execution.

| <b>or-m</b> | tt | ff | ee |
|-------------|----|----|----|
| tt          | tt | tt | tt |
| ff          | tt | ff | ee |
| ee          | ee | ee | ee |

| <b>and-m</b> | tt | ff | ee |
|--------------|----|----|----|
| tt           | tt | ff | ee |
| ff           | ff | ff | ff |
| ee           | ee | ee | ee |

| <b>not-m</b> |    |
|--------------|----|
| tt           | ff |
| ff           | tt |
| ee           | ee |

Tab. 2.10-1 Propositional operators of John McCarthy

To see the intuition behind McCarthy's operators consider the expression  $p \text{ or-m } q$  assuming that its arguments are computed from left to right<sup>10</sup>:

- If  $p = tt$ , then we give up the evaluation of  $q$  (lazy evaluation) and assume that the value of the expression is  $tt$ . Notice that in this case we possibly avoid entering an infinite computation.
- If  $p = ff$ , then we evaluate  $q$ , and its value becomes the value of the expression; it also means that if we enter an infinite execution, we remain in it. Here we also possibly avoid entering an infinite computation.
- If  $p = ee$ , then this means that the evaluation aborts or loops at the evaluation of  $p$ , hence  $q$  will never be evaluated. As a consequence, the final value of our expression must be the value of  $p$ . Note that in this place the infiniteness is "signalized" by itself, and, therefore, its non-decidability doesn't cause a problem.

The rule for **and** is analogous. It is to be emphasized that McCarthy's operators coincide with classical operators on classical values (grey fields in the table).

McCarthy's implication is defined in a classical way, i.e. by a combination of alternative and negation operators:

$$p \text{ implies-m } q = (\text{not-m } p) \text{ or-m } q$$

It is to be noted that not all classical tautologies remain satisfied in McCarthy's calculus. Among those that are satisfied we have<sup>11</sup>:

- associativity of disjunction and conjunction,
- De Morgan's laws

and among the non-satisfied are:

- **or-m** and **and-m** are not commutative, e.g.,  $ff \text{ and-m } ee = ff$  but  $ee \text{ and-m } ff = ee$ ,
- **and-m** is distributive over **or-m** only on the right-hand side, i.e.

$$\begin{aligned}
 p \text{ and-m } (q \text{ or-m } s) &= (p \text{ and-m } q) \text{ or-m } (p \text{ and-m } s) \text{ however} \\
 (q \text{ or-m } s) \text{ and-m } p &\neq (q \text{ and-m } p) \text{ or-m } (s \text{ and-m } p) \text{ since} \\
 (tt \text{ or-m } ee) \text{ and-m } ff &= ff \text{ and } (tt \text{ and-m } ff) \text{ or-m } (ee \text{ and-m } ff) = ee
 \end{aligned}$$

- analogously **or-m** is distributive over **and-m** only on the right-hand side,
- $p \text{ or-m } (\text{not-m } p)$  does not need to be true but is never false,
- $p \text{ and-m } (\text{not-m } p)$  does not need to be false but is never true.

On the ground of McCarthy's calculus, we define in Sec. 6.4.1 the denotations of three-valued partial boolean expressions.

An alternative to McCarthy's propositional calculus is that of Kleene with operators defined in the following way:

<sup>10</sup> The suffix "-m" stands for "McCarthy" and is used to distinguish McCarthy's operators not only from classical ones but also from the operators of Kleene, which are discussed later.

<sup>11</sup> This claim is true only in the case of a single error element.



| or-k | tt | ff | ee | and-k | tt | ff | ee | not-k |    |
|------|----|----|----|-------|----|----|----|-------|----|
| tt   | tt | tt | tt | tt    | tt | ff | ee | tt    | ff |
| ff   | tt | ff | ee | ff    | ff | ff | ff | ff    | tt |
| ee   | tt | ee | ee | ee    | ee | ff | ee | ee    | ee |

Tab. 2.10-2 Propositional operators of Steven Kleene

In this case

$$\begin{aligned} \text{tt or-k ee} &= \text{ee or-k tt} = \text{tt} \\ \text{ff and-k ee} &= \text{ee and-k ff} = \text{ff} \end{aligned}$$

In Kleene's calculus whenever any argument of **or-k** is **tt**, then the result is **tt**, and analogously for **and-k**. Due to this assumption, we gain commutativity of both operators, but if we want to evaluate boolean expressions in this way, we had to compute both arguments of our operators “in parallel”. Since that would be hardly acceptable, we use McCarthy's calculus in boolean expressions. If, however, we use a propositional calculus in proofs rather than in computations, then Kleene's calculus is more convenient. This is why we shall use it in conditions that describe properties of programs (see Sec. 9.2).

## 2.11 Data algebras

Datatypes that are used in programs — such as integers, booleans, strings, arrays, lists, etc. — are usually associated with some operations on them. For instance, a data type of integers may be associated with the following arithmetical operations, and comparison predicates:

$$\begin{aligned} \text{plus} &: \text{IntegerE} \times \text{IntegerE} \mapsto \text{IntegerE} \\ \text{minus} &: \text{IntegerE} \times \text{IntegerE} \mapsto \text{IntegerE} \\ \text{times} &: \text{IntegerE} \times \text{IntegerE} \mapsto \text{IntegerE} \\ \text{divide} &: \text{IntegerE} \times \text{IntegerE} \mapsto \text{IntegerE} \\ \text{less} &: \text{IntegerE} \times \text{IntegerE} \mapsto \text{BooleanE} \\ \text{equal} &: \text{IntegerE} \times \text{IntegerE} \mapsto \text{BooleanE} \end{aligned} \tag{2.11-1}$$

where

$$\begin{aligned} \text{int} &: \text{IntegerE} = \text{Integer} \mid \text{Error} \\ \text{boo} &: \text{BooleanE} = \{\text{tt}, \text{ff}\} \mid \text{Error} \end{aligned}$$

and where **Integer** is a set of integers representable in a current implementation.

A mathematical being that includes some sets and operations on them is called a *many-sorted algebra*. The sets in the algebra are called its *sorts*, and functions — its *constructors*.

As we are going to see later, an algebra of data usually includes more than one sort, and with each sort it includes some constructors. In our case we may add the following constructors of booleans:

$$\begin{aligned} \text{or} &: \text{BooleanE} \times \text{BooleanE} \mapsto \text{BooleanE} \\ \text{and} &: \text{BooleanE} \times \text{BooleanE} \mapsto \text{BooleanE} \\ \text{not} &: \text{BooleanE} \mapsto \text{BooleanE} \end{aligned}$$

Additionally, we may wish to add to our algebra zero-argument constructors that build some data “out of nothing”:

$$\begin{aligned} \text{create-zero} &: \mapsto \text{IntegerE} \\ \text{create-one} &: \mapsto \text{IntegerE} \\ \text{create-true} &: \mapsto \text{BooleanE} \\ \text{create-false} &: \mapsto \text{BooleanE} \end{aligned}$$

Such zero-argument operations are called *constants*. We may need constants in an algebra if we want our algebra to have *reachable elements*, i.e. elements generable by constructors. Note that without integer constants we can't



generate “the first integer”. In turn, to do so we need only one constant **create-one**, since once we have 1 we can generate all other integers and booleans.

Sometimes, for technical reasons, we may wish to have “superfluous” constants, although usually not in the algebra of data. We will see such situations when building the algebras of denotations in Sec. 6.

Many-sorted algebras may be visualized graphically as in Fig. 2.11-1. For simplicity we included only some operation of the algebra and used shorter names of constructors

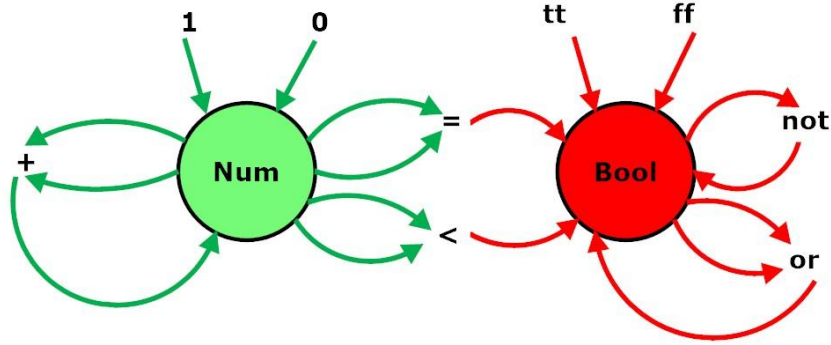


Fig. 2.11-1 Graphical representation of a two-sorted algebra

## 2.12 Many-sorted algebras

Our algebra discussed in Sec. 2.11, let's call it **AlgIntBool**, is a two-sorted algebra and constitutes a particular case of many-sorted algebras. Such algebras will constitute one of our main tools in building denotational models, and therefore, we shall briefly introduce their theory in this section and sections 2.13, 2.14, and 2.15. Since this part of our book has an abstract mathematical character, we shall return for a while to traditional typesetting of indices as  $a_i$  rather than  $a.i$ .

By a many-sorted algebra we shall mean a tuple:

$$\mathbf{Alg} = (\text{Sig}, \text{Car}, \text{Fun}, \text{car}, \text{fun})$$

where

|   |  |
|---|--|
| $\text{Sig} = (\text{Cn}, \text{Fn}, \text{ar}, \text{so})$ | — is called the <i>signature of the algebra</i> ,  |
| $\text{Cn}$   | — is a finite set of words called the <i>names of carriers</i> ; these names are usually called the <i>sorts of the algebra</i> ,  |
| $\text{Fn}$   | — is a finite set of words that are the <i>names of functions</i> ; the functions themselves are called <i>constructors</i>  |
| $\text{ar} : \text{Fn} \mapsto \text{Cn}^{c*}$              | — with every name of a function $\text{fn}$ there is associated a finite (possibly empty) sequence of sorts<br>$\text{ar}.\text{fn} = (\text{cn}_1, \dots, \text{cn}_k)$<br>called the <i>arity</i> of $\text{fn}$ <sup>12</sup> |
| $\text{so} : \text{Fn} \mapsto \text{Cn}$                   | — to every name of a function $\text{fn}$ the function $\text{so}$ assigns a carrier name $\text{so}.\text{fn}$ which is called <i>the sort of <math>\text{fn}</math></i> ,  |
| $\text{Car}$  | — a finite set of <i>carriers</i> ,  |

<sup>12</sup> The word „arity” comes from unary, binary, ternary etc.

|   |   |
|---|---|
| $\mathbf{Fun}$                                    | — a finite set of total functions with arguments and values in carriers; these functions are called <i>constructors</i> ,   |
| $\mathbf{car} : \mathbf{Cn} \mapsto \mathbf{Car}$ | — to every name $\mathbf{cn}$ of a carrier function $\mathbf{car}$ assigns a corresponding carrier $\mathbf{car.cn}$ ,  |
| $\mathbf{fun} : \mathbf{Fn} \mapsto \mathbf{Fun}$ | — to every function name $\mathbf{fn}$ such that<br>$\mathbf{ar.fn} = (\mathbf{cn}_1, \dots, \mathbf{cn}_k)$<br>$\mathbf{so.fn} = \mathbf{cn}$<br>the function $\mathbf{fun}$ assigns a total function<br>$\mathbf{fun.fn} : \mathbf{car.cn}_1 \times \dots \times \mathbf{car.cn}_k \mapsto \mathbf{car.cn}$ |

The concepts of *arity* and *sort* are applied not only to function names but also to the corresponding functions. Functions in the set  $\mathbf{Fun}$  are traditionally called *constructors*. The tuple  $((\mathbf{cn}_1, \dots, \mathbf{cn}_k), \mathbf{cn})$  that describes the arity and the sort of a constructor will be called the *signature* of that constructor.

Zero-argument constructors, i.e., constructors whose arity is an empty sequence, are called *constants* of the algebra. If  $\mathbf{f}$  is such a constant, then we write

$$\mathbf{f} : \mapsto \mathbf{Carrier}$$

and the unique value of  $\mathbf{f}$  is written as

$$\mathbf{f}().$$

It should be emphasized that all constructors of an algebra are total functions. In our approach this is possible due to the use of abstract errors (Sec. 2.9).

As we will see in the sequel, the signatures of many-sorted algebras have been introduced to describe the derivation of syntax from denotations in constructing programming languages. For concrete algebras, e.g., such as discussed in Sec.2.11, the signature is implicit in a corresponding set of formulas (2.11-1).

Two many-sorted algebras are said to be *similar* if they have the same signature. In the future, we shall frequently define concrete algebras by defining their carriers and constructors but without showing their signatures explicitly. In that case, we shall say that two algebras are similar if it is possible to construct a common signature for them.

Consider two algebras

$$\mathbf{Alg}_i = (\mathbf{Sig}_i, \mathbf{Car}_i, \mathbf{Fun}_i, \mathbf{car}_i, \mathbf{fun}_i) \quad \text{for } i = 1, 2$$

with signatures

$$\mathbf{Sig}_i = (\mathbf{Cn}_i, \mathbf{Fn}_i, \mathbf{ar}_i, \mathbf{so}_i) \quad \text{for } i = 1, 2$$

We say that  $\mathbf{Sig}_2$  is an *extension* of  $\mathbf{Sig}_1$  or that  $\mathbf{Sig}_1$  is a *restriction* of  $\mathbf{Sig}_2$ , if

1.  $\mathbf{Cn}_1 \subseteq \mathbf{Cn}_2$  and  $\mathbf{Fn}_1 \subseteq \mathbf{Fn}_2$ ,
2. functions  $\mathbf{ar}_2, \mathbf{so}_2$  coincide with  $\mathbf{ar}_1, \mathbf{so}_1$  on  $\mathbf{Fn}_1$ .

We say that algebra  $\mathbf{Alg}_2$  is an *extension* of algebra  $\mathbf{Alg}_1$ , if

1.  $\mathbf{Sig}_2$  is an extension of  $\mathbf{Sig}_1$ ,
2.  $\mathbf{car}_1.\mathbf{cn} \subseteq \mathbf{car}_2.\mathbf{cn}$  for every sort  $\mathbf{cn} : \mathbf{Cn}_1$ ,
3.  $\mathbf{fun}_2.\mathbf{fn}$  coincides with  $\mathbf{fun}_1.\mathbf{fn}$  on the appropriate carriers for every  $\mathbf{fn} : \mathbf{Fn}_1$ .

In other words, each (nontrivial) extension of an algebra results from that algebra by adding new carriers and/or new constructors and/or new elements to the existing carriers.

If  $\mathbf{Alg}_1$  and  $\mathbf{Alg}_2$  are similar, then we say that  $\mathbf{Alg}_1$  is a *subalgebra* of  $\mathbf{Alg}_2$  if:

1. the carriers of  $\mathbf{Alg}_1$  are subsets of the corresponding carriers of  $\mathbf{Alg}_2$ ,

2. the constructors of **Alg<sub>1</sub>** coincide with constructors of **Alg<sub>2</sub>** on the carriers of **Alg<sub>1</sub>**.

For every algebra there exists its unique subalgebra (maybe empty), called the *reachable subalgebra*, that includes only these elements of the algebra that can be generated by its constructors. If an algebra is identical with its reachable subalgebra, then it is said to be *reachable*.

Another important concept associated with many-sorted algebra are *many-sorted* homomorphism between them. By a *many-sorted homomorphism* from algebra **Alg<sub>1</sub>** into a similar algebra **Alg<sub>2</sub>** where we call a family of functions

$$H = \{h.cn \mid cn : Cn\}$$

whose elements — called *the components of that homomorphism* — map the elements of **Alg<sub>1</sub>** into the elements of **Alg<sub>2</sub>**, hence

$$h.cn : car_1.cn \mapsto car_2.cn \quad \text{for all } cn : Cn$$

and where for every constructor name  $fn : Cn$  such that

$$ar.fn = (cn_1, \dots, cn_n) \quad \text{where } n \geq 0$$

and every tuple of arguments

$$(a_1, \dots, a_n) : car_1.cn_1 \times \dots \times car_1.cn_n$$

the following equality holds

$$h.cn.(fun_1.fn.(a_1, \dots, a_n)) = fun_2.fn.(h.cn_1.a_1, \dots, h.cn_n.a_n) \quad (2.12-1)$$

In other words a homomorphic image of the value of a function  $fun_1.fn$  from the first algebra with arguments  $(a_1, \dots, a_n)$  equals the value of the corresponding function  $fun_2.fn$  from the second algebra applied to the tuple of homomorphic images of the first tuple i.e. applied to  $(h.cn_1.a_1, \dots, h.cn_n.a_n)$ . Notice that for  $n = 0$  the equality (2.12-1) has the form

$$h.cn.(fun_1.fn.()) = fun_2.fn.()$$

The fact that  $H$  is a homomorphism from **Alg<sub>1</sub>** into **Alg<sub>2</sub>** shall be written as:

$$H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2$$

In the general case, homomorphisms do not map algebras onto algebras but into algebras, which means that not every element in **Alg<sub>2</sub>** must be an image of an element from **Alg<sub>1</sub>**. For instance an identity homomorphism from integers to numbers

$$I2N : (\text{Integer}, 1, \text{plus}, \text{minus}) \mapsto (\text{Number}, 1, \text{plus}, \text{minus})$$

is not “onto”, whereas a homomorphism from integers into even integers

$$I2E : (\text{Integer}, 1, \text{plus}, \text{minus}) \mapsto (\text{Even}, 1, \text{plus}, \text{minus})$$

defined by the equality  $I2E.int = 2 * int$  is “onto”. In the general case a homomorphism  $H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2$  is called:

- a *monomorphism* — if all its components are one-to-one functions; e.g.,  $I2N$  and  $I2E$ ,
- an *epimorphism* — if all its components are “onto”; e.g.,  $I2E$
- an *isomorphism* — if it is both a monomorphism and an epimorphism; e.g.,  $I2E$ .

**Theorem 2.12-1** For every homomorphism  $H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2$ , the image of **Alg<sub>1</sub>** in **Alg<sub>2</sub>**, i.e., the restriction of **Alg<sub>2</sub>** to the images through  $H$  of **Alg<sub>1</sub>** with the appropriate truncation of constructors of **Alg<sub>2</sub>** constitutes a subalgebra of **Alg<sub>2</sub>**. ■

**Proof** To prove our theorem, we have to show that the images in **Alg<sub>2</sub>** of the carriers of **Alg<sub>1</sub>** are closed under the operations of **Alg<sub>2</sub>**. Let then  $(b_1, \dots, b_n)$  from **Alg<sub>2</sub>**, be the image of  $(a_1, \dots, a_n)$  in **Alg<sub>1</sub>**, i.e. let:

$$(b_1, \dots, b_n) = (h.cn_1.a_1, \dots, h.cn_n.a_n)$$

Let furthermore for some function name  $fn$

$$\text{fun}_2.\text{fn}.(b_1, \dots, b_n) = b$$

We have to show that  $b$  has a coimage in  $\text{Alg}_1$ . It is indeed the case since on the ground of (2.12-1):

$$\text{fun}_2.\text{fn}.(b_1, \dots, b_n) = \text{fun}_2.\text{fn}.(h.\text{cn}_1.a_1, \dots, h.\text{cn}_n.a_n) = h.\text{cn}.(f_{un}_1.\text{fn}.(a_1, \dots, a_n))$$

hence  $h.\text{cn}.(f_{un}_1.\text{fn}.(a_1, \dots, a_n))$  is the coimage of  $b$  in  $\text{Alg}_1$ . ■

An algebra, which is the image of a homomorphism,  $\text{Alg}_1 \mapsto \text{Alg}_2$  is called *the kernel of the homomorphism*  $H$  in  $\text{Alg}_2$ .

All our investigations about homomorphisms can be generalized to the case where the signatures of two algebras

$$\text{Sig}_i = (\text{Cn}_i, \text{Fn}_i, \text{ar}_i, \text{so}_i) \quad \text{for } i = 1, 2$$

are not identical but are *similar* in the sense that there exist two reversible functions of similarity

$$\text{S}_n : \text{Cn}_1 \mapsto \text{Cn}_2$$

$$\text{S}_f : \text{Fn}_1 \mapsto \text{Fn}_2$$

such that if

$$\text{S}_f.\text{fn}_1 = \text{fn}_2$$

$$\text{ar}_1.\text{fn}_1 = \text{cn}_{11}, \dots, \text{cn}_{1p}$$

$$\text{ar}_2.\text{fn}_2 = \text{cn}_{21}, \dots, \text{cn}_{2m}$$

then

$$p = m$$

$$\text{S}_n.\text{cn}_{1i} = \text{cn}_{2i} \quad \text{for } i = 1; p$$

In other words, two signatures are similar if they have the same number of carrier names and function names, and the corresponding function names have identical arities and sorts up to the names of carriers.

Now we can generalize the notion of the similarity of algebras: two algebras shall be called *similar* if their signatures are similar. For any fixed functions,  $\text{S}_n$  and  $\text{S}_f$  the concept of homomorphism, and the corresponding theorems remain valid for the generalized similarity.

## 2.13 Abstract syntax

Every signature

$$\text{Sig} = (\text{Cn}, \text{Fn}, \text{ar}, \text{so})$$

unambiguously determines a certain algebra with that signature and with formal languages as carriers. This algebra is called *abstract syntax over signature*  $\text{Sig}$  and will be denoted by  $\text{AbsSy}(\text{Sig})$ <sup>13</sup>. The elements of its carriers are words of a many-sorted formal language

$$\{\text{Lan.cn} \mid \text{cn} : \text{Cn}\}$$

defined by an equational grammar (see Sec.2.6) in a way described below.

To every carrier name  $\text{cn}$  we associate a language denoted by  $\text{Lan.cn}$ . The family (tuple) of all these languages is defined by an equational grammar where for every  $\text{cn} : \text{Cn}$  we have the following equation<sup>14</sup>:

<sup>13</sup> The idea of an abstract syntax regarded as a mathematical idealization of a syntax of a programming language appeared for the first time in papers of J. McCarthy [78] and P. Landin [66]. Abstract syntax was associated with abstract algebras by J.A. Goguen, J.W. Thatcher, E.G. Wagner and J.B. Wright [58]. A little later A.Blikle [29] used that concept in an attempt to give a formal semantics to a subset of Pascal. In his paper abstract syntax was technically understood in a slightly different way than here, but the idea was roughly the same.

<sup>14</sup> We assume, of course, that the commas “,” and the parentheses “(” and “)” do not appear in the signature as constructors’ names.

$$\begin{aligned} \text{Lan.cn} &= \{fn_1\} \odot \{\{\} \odot \text{Lan.cn}_{11} \odot \{,\} \odot \dots \odot \{,\} \odot \text{Lan.cn}_{1n(1)} \odot \{\}\} | \\ &\dots \\ &\{fn_k\} \odot \{\{\} \odot \text{Lan.cn}_1 \odot \{,\} \odot \dots \odot \{,\} \odot \text{Lan.cn}_{n(k)} \odot \{\}\} \end{aligned} \quad (2.13-1)$$

Here  $fn_i$  for  $i = 1;k$  are function names with

$$\text{so.fn}_i = \text{cn}$$

and

$$\text{ar.fn}_i = (\text{cn}_{i1}, \dots, \text{cn}_{in(i)}) \quad \text{for } i = 1;k$$

We assume that if for a carrier name  $\text{cn}$  there is no function name  $fn$  such that  $\text{so.nf} = \text{cn}$ , then the corresponding language is empty, i.e. its defining equation is:

$$\text{Lan.cn} = \{\}$$

For every non-empty  $\text{Lan.cn}$ , its elements are words of the form

$$fn_i(w_{i1}, \dots, w_{in(i)})$$

i.e. of the form  $fn_i \odot ( \odot w_{i1} \odot \dots \odot w_{in(i)} \odot )$  where  $\odot$  is the concatenation of words and

$$w_{ik} : \text{Lan.cn}_k.$$

As is easy to see, for every algebra **Alg** its abstract syntax algebra is reachable, although it may be empty if there are no constants in **Alg**.

Since abstract syntaxes are generated from signatures, they may be associated with arbitrary algebras (through their signatures). If **Alg** is an algebra with signature **Sig**, then **AbsSy(Sig)** will be called *the abstract syntax of algebra Alg*. For instance, if **AlgIntBoo** is the two-sorted algebra described in Sec.2.1.1 then the carrier of its abstract syntax are defined by the following equational grammar, where **IntExp** and **BooExp** are languages of integer expressions and boolean expressions respectively:

$$\text{IntExp} = \quad (2.13-1)$$

```

0
1
plus(IntExp, IntExp)
minus(IntExp, IntExp)
times(IntExp, IntExp)
divide(IntExp, IntExp)

```

$$\text{BooExp} =$$

```

tt
ff
less(IntExp, IntExp)
equal(IntExp, IntExp)
or(BooExp, BooExp)
and(BooExp, BooExp)
not(BooExp)

```

In this grammar, we use four notational conventions that we shall assume as standards for future use (cf. Sec. 2.1.1):

1. as already announced in Sec. 2.1.1, characters and words such as **0**, **1**, **plus**, **(, )** etc. that appear at the level of syntax are typeset in **Arial Narrow**, whereas **IntExp** and **BooExp** are typeset in **Arial**, since they are metavariables from the level of **MetaSoft**,
2. one-element sets are identified with their elements, i.e. instead of  $\{a\}$  we write **a**,
3. the values of zero-argument constructors are written without the empty tuples of arguments, i.e. we write **1** instead of **1.()**.

4. the concatenation sign  $\odot$  is omitted, e.g., instead of  $a \odot b$  we write  $a b$ ,

Examples of a numeric and a boolean abstract-syntax expressions written in this style are the following:

- $\text{plus}(\text{plus}(\text{minus}(1,0),1),\text{plus}(1,1))$
- $\text{or}(\text{less}(\text{plus}(\text{plus}(\text{minus}(1,0),1),\text{plus}(1,1)),\text{plus}(1,1)),\text{ff})$

As we see, the expressions of our languages do not contain variables and are written in a *prefix notation* where function symbols always precede their arguments. E.g., we write  $\text{plus}(1,1)$  instead of  $(1 \text{ plus } 1)$ . The latter style is called *infix-notation*.

In the syntactic algebra defined by our grammar, the elements of carriers are numeric and boolean expressions, respectively (without variables), and constructors correspond to constructor names from our signature. For instance, with a constructor name  $\text{plus}$ , we associate a constructor  $[\text{plus}]$  of the algebra  $\mathbf{AbsSy}(\text{Sig})$  defined by the equation

$$[\text{plus}].[\text{num-exp}_1, \text{num-exp}_2] = \text{plus}(\text{num-exp}_1, \text{num-exp}_2)^{15}$$

This constructor, given two expressions  $\text{num-exp}_1$  and  $\text{num-exp}_2$  returns the expression of the form  $\text{plus}(\text{num-exp}_1, \text{num-exp}_2)$ . E.g. given  $\text{times}(x,y)$  and  $\text{plus}(z,y)$  returns

$$\text{plus}(\text{times}(x,y), \text{plus}(x,y))$$

Now we can formulate a theorem which is fundamental for denotational models of programming languages.

**Theorem 2.13-1** For every many-sorted algebra  $\mathbf{Alg}$  with a signature  $\text{Sig}$  there is exactly one homomorphism  $H : \mathbf{AbsSy}(\text{Sig}) \mapsto \mathbf{Alg}$ . ■

**Proof** Every homomorphism  $H : \mathbf{AbsSy}(\text{Sig}) \mapsto \mathbf{Alg}$  must (from the definition) satisfy the equation:

$$H.\text{cn}.[\text{fn}(w_1, \dots, w_n)] = \text{fun}.\text{fn}.[H.\text{cn}_1.w_1, \dots, H.\text{cn}_n.w_n]$$

where

$$\begin{aligned} \text{ar}.\text{fn} &= (\text{cn}_1, \dots, \text{cn}_n) \\ \text{so}.\text{fn} &= \text{cn} \\ w_i &: \text{Lan}.\text{cn}_i \quad \text{for } i = 1; n \end{aligned}$$

Since every word in abstract syntax is of a unique (for it) form  $\text{fn}(w_1, \dots, w_n)$ , the above equations (for all  $\text{fn}$ ) define the family  $\{H.\text{cn} \mid \text{cn} : \text{Cn}\}$  in an unambiguous way. In the case of empty carriers of  $\mathbf{AbsSy}(\text{Sig})$  the corresponding components of  $H$  are empty. ■

The unique homomorphism from  $\mathbf{AbsSy}(\text{Sig})$  to  $\mathbf{Alg}$  will be called *the semantics of abstract syntax*. For instance, if by  $\{\text{In}, \text{Bo}\}$  we denote the semantics of abstract syntax of  $\mathbf{AlgIntBoo}$ , then this homomorphism maps boolean expression  $\text{less}(\text{plus}(1,1), \text{times}(1,1))$  into boolean value  $\text{ff}$ :

$$\begin{aligned} \text{Bo}.[\text{less}(\text{plus}(1,1), \text{times}(1,1))] &= \\ \text{fun}.\text{less}.\text{(No.}[\text{plus}(1,1)], \text{No.}[\text{times}(1,1)]) &= \\ \text{fun}.\text{less}.\text{(fun.plus.(No.}[\text{1}], \text{No.}[\text{1}]), \text{fun.times.}([\text{No.}[\text{1}], \text{No.}[\text{1}]))} &= \\ \text{fun.less (fun.plus(1,1), fun.times(1,1))} &= \\ \text{ff} \end{aligned}$$

On the ground of theorems 2.12-1 and 2.13-1, in every algebra  $\mathbf{Alg}$ , there is a unique subalgebra which is the kernel of the semantics of abstract syntax of  $\mathbf{Alg}$ . That algebra is the (unique) reachable subalgebra of  $\mathbf{Alg}$ . For instance, the reachable subalgebra of the algebra

$$(\text{RealE}, 1, \text{plus}, \text{divide})$$

is the algebra of positive rational numbers

<sup>15</sup> The meta-parentheses “[“ and “]” are introduced in order to distinguish them from parentheses that belong to the defined language.

(PosRat, 1, plus, divide)

since only such numbers can be constructed from 1 in using **plus** and **divide**. Notice that if we remove 1 from this algebra, then its reachable subalgebra becomes empty and consequently its algebra of abstract syntax will be empty as well.

**Theorem 2.13-2** For any two similar algebras **Alg<sub>1</sub>** and **Alg<sub>2</sub>**, if **Alg<sub>1</sub>** is reachable, then there is at most one homomorphism

$$H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2,$$

and if this is the case, then the image of **Alg<sub>1</sub>** in **Alg<sub>2</sub>** is reachable. ■

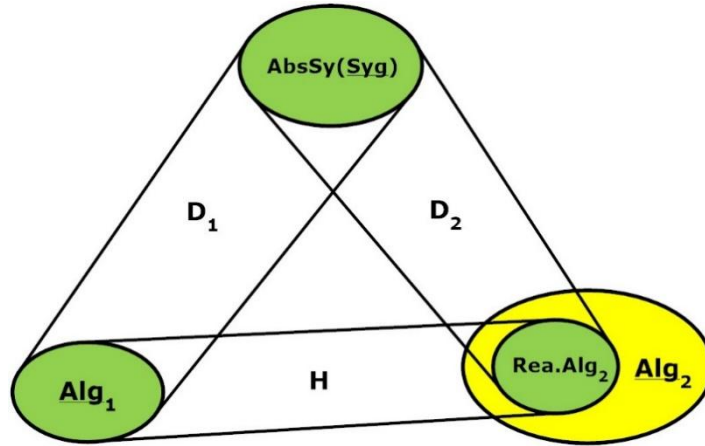


Fig. 2.13-1 Reachable algebras

**Proof.** The theorem and its proof are illustrated in Fig. 2.13-1. Since **Alg<sub>1</sub>** and **Alg<sub>2</sub>** are similar, they must have a common signature **Sig** and a common abstract syntax **AbsSy(Sig)**. Therefore — on the ground of Theorem 2.13-1 — there exist two unambiguously defined semantics of abstract syntaxes

$$\begin{aligned} D_1 &: \mathbf{AbsSy}(\mathbf{Sig}) \mapsto \mathbf{Alg}_1 \text{ and} \\ D_2 &: \mathbf{AbsSy}(\mathbf{Sig}) \mapsto \mathbf{Alg}_2 \end{aligned}$$

Now, if there exists a homomorphism  $H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2$ , then the composition

$$D_1 \bullet H : \mathbf{AbsSy}(\mathbf{Sig}) \mapsto \mathbf{Alg}_2$$

defined as the composition of their components is a homomorphism. Since  $D_2$  is the unique homomorphism between these algebras, we have

$$D_1 \bullet H = D_2,$$

and since **Alg<sub>1</sub>** is reachable, the above equation defines  $H$  unambiguously, because otherwise, we could define another homomorphism from **AbsSy(Sig)** into **Alg<sub>2</sub>** which would contradict Theorem 2.13-1. This proves that the image of **Alg<sub>1</sub>** in **Alg<sub>2</sub>** is reachable. ■

As an immediate consequence of this theorem we have another theorem:

**Theorem 2.13-3** For every nonempty algebra **Alg** over signature **Sig** the following claims are equivalent:

- (1) **Alg** is reachable,
- (2) every homomorphism of the type  $H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}$  (for an arbitrary **Alg<sub>1</sub>**) is onto,
- (3) the semantics of abstract syntax  $D : \mathbf{AbsSy}(\mathbf{Sig}) \mapsto \mathbf{Alg}$  is onto. ■

**Proof** Let **Alg** be reachable and let for some **Alg<sub>1</sub>** similar to **Alg** there exist a homomorphism

$$H : \mathbf{Alg}_1 \mapsto \mathbf{Alg},$$

and let

$$D : \mathbf{AbsSy}(\mathbf{Sig}) \mapsto \mathbf{Alg}_1$$



be the abstract-syntax semantics of **Alg**<sub>1</sub>. In that case

$$D \bullet H : \mathbf{AbsSy}(\text{Sig}) \mapsto \mathbf{Alg}$$

is the abstract-syntax semantics for **Alg**, hence, since **Alg** is reachable, then  $D \bullet H$  must be *onto*, and therefore also  $H$  must be *onto*. Hence (1) implies (2). Now (3) follows from (2) as its particular case, and (2) implies (1) by the definition of reachability. ■

At the end of this section, one more useful theorem:

**Theorem 2.13-4** An algebra has a nonempty reachable subalgebra if and only if it contains at least one zero-argument constructor. ■

**Proof** If there is a constant in the algebra, then it belongs to its reachable part, and hence, this part is not empty. If, however, such a constant does not exist, then in the grammar corresponding to that algebra, there are no constant monomials, and therefore all the carriers of abstract syntax are empty. Therefore the reachable part of **Alg** is an empty algebra. ■

Abstract syntaxes are, in general, not very convenient in practical programming, and therefore they are usually replaced by more user-friendly syntaxes historically called *concrete syntaxes*. In such a case, elements of abstract syntax correspond to *parsing trees* of concrete scripts (see, e.g. [3]).

## 2.14 Ambiguous and unambiguous algebras

An algebra **Alg** with a signature **Sig** is said to be *unambiguous* if its abstract-syntax semantics

$$D : \mathbf{AbsSy}(\text{Sig}) \mapsto \mathbf{Alg}$$

is a monomorphism, i.e., if for every carrier **Car.cn** of **Alg** and every element **e** of that carrier there is at most one word  $w : \text{Lan.cn}$  in the abstract syntax  $\mathbf{AbsSy}(\text{Sig})$  such that

$$D.\text{cn}.w = e$$

Algebras which are not unambiguous will be called *ambiguous*.

Algebras of denotations of programming languages are practically always ambiguous. For instance, the algebra **AlgIntBoo** described in 2.11 is ambiguous since, e.g., two different words  $\text{plus}(\text{plus}(1,1),1)$  and  $\text{plus}(1,\text{plus}(1,1))$  correspond to the same number 3.

Now consider two algebras **Alg**<sub>1</sub> and **Alg**<sub>2</sub> with a common signature **Sig** hence also with a common abstract syntax **SkAbs**(**Sig**). Let

$$D_1 : \mathbf{SkAbs}(\text{Sig}) \mapsto \mathbf{Alg}_1$$

$$D_2 : \mathbf{SkAbs}(\text{Sig}) \mapsto \mathbf{Alg}_2$$

be two corresponding abstract-syntax semantics. Algebra **Alg**<sub>1</sub> is said to be *less (or equally) ambiguous than* algebra **Alg**<sub>2</sub>, that we shall write as

$$\mathbf{Alg}_1 \preceq \mathbf{Alg}_2$$

if the homomorphism  $D_2$  is gluing not more than  $D_1$  (Fig. 2.14-1), i.e., if for any two words  $w_1$  and  $w_2$  in abstract syntax that belong to the same carrier **Car.cn** the following implication holds:

$$\text{if } D_1.\text{cn}.w_1 = D_1.\text{cn}.w_2 \quad \text{then} \quad D_2.\text{cn}.w_1 = D_2.\text{cn}.w_2$$

Intuitively speaking, whenever an element of **Alg**<sub>1</sub> may be constructed in two different ways, the two ways lead to the same element in **Alg**<sub>2</sub>.



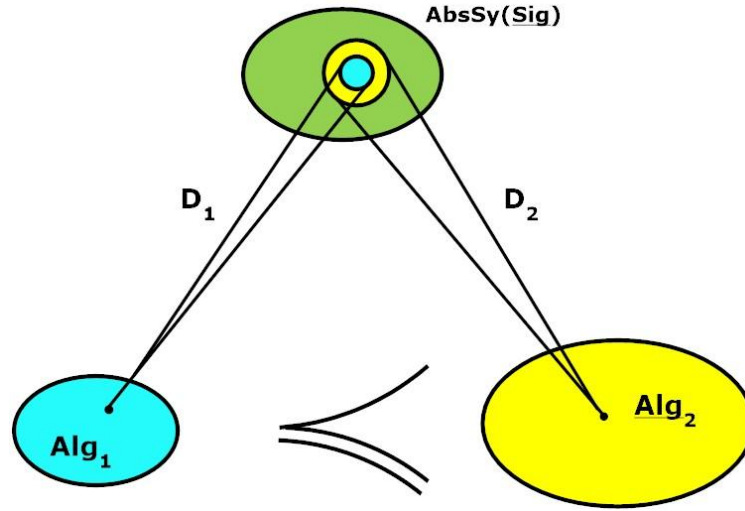


Fig. 2.14-1 Two ambiguous algebras

Ambiguous algebras play an important role in the theory of programming languages since, for the majority of existing languages, their algebras of concrete syntax — if formally described — would turn out to be ambiguous. To explain this fact assume that **AbsSy(Sig)** is defined by the grammar

$$\text{IntExp} = 0 \mid 1 \mid +(\text{IntExp}, \text{IntExp}),$$

**Alg<sub>1</sub>** is an algebra of infix expressions without parentheses defined by the grammar

$$\text{IntExp} = 0 \mid 1 \mid \text{IntExp} + \text{IntExp}$$

and **Alg<sub>2</sub>** is the algebra of integers. Let now **D<sub>1</sub>** replaces prefixes by infixes and removes parentheses.

Anticipating the considerations of Sec. 3, the algebra of numbers is the *algebra of denotations* (of meanings) for both our algebras of numeric expressions and the homomorphism **D<sub>2</sub>** is the *denotational homomorphism* (the *semantics*) of the algebra of abstract syntax. Now, we may ask, if there exists a denotational homomorphism

$$\mathbf{D}_{12} : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2$$

from parentheses-free expressions into numbers.

To answer this question notice that for such algebras and their corresponding homomorphisms the following equalities hold:

$$\mathbf{D}_1.[+(+(1,1),1)] = 1+1+1 \quad \mathbf{D}_2.[+(+(1,1),1)] = 3$$

$$\mathbf{D}_1.[+(1,+(1,1))] = 1+1+1 \quad \mathbf{D}_2.[+(1,+(1,1))] = 3$$

As we see **D<sub>1</sub>** is gluing not more than **D<sub>2</sub>**. In “practical mathematics”, hence also in programming languages, we frequently omit “unnecessary parentheses” when we deal with associative operations. The corresponding algebras are, in general, ambiguous, and therefore, the denotational homomorphism **D<sub>12</sub>** need not exist. If however, they are not more ambiguous than the algebras of denotations, then such a homomorphism exist which follows from the following theorem:

**Theorem 2.14-1** If **Alg<sub>1</sub>** and **Alg<sub>2</sub>** are similar and **Alg<sub>1</sub>** is reachable, then the (unique) homomorphism

$$\mathbf{D}_{12} : \mathbf{Alg}_1 \mapsto \mathbf{Alg}_2 \text{ exists iff } \mathbf{Alg}_1 \preceq \mathbf{Alg}_2. \blacksquare$$

This unique homomorphism may be constructed as (intuitively speaking) the composition of the inverse of **D<sub>1</sub>** with **D<sub>2</sub>**, i.e.

$$\mathbf{D}_{12} = \mathbf{D}_1^{-1} \bullet \mathbf{D}_2.$$

Although the inverse of **D<sub>1</sub>** maps the elements of **Alg<sub>1</sub>** into sets of abstract expressions, yet all these expressions are mapped by **D<sub>2</sub>** into the same element of **Alg<sub>2</sub>**. For a formal proof of this theorem, see [32].

Of course, if  $D_1$  is an isomorphism then  $\mathbf{Alg}_1$  is “equally ambiguous” as  $\mathbf{Alg}_2$ , and therefore the homomorphism  $D_{12}$  exists.

## 2.15 Algebras and grammars

The first step in the process of programming-language construction consists in defining an algebra of denotations from which we derive a unique algebra of abstract syntax. Since the latter is usually not user-friendly, we transform it into a concrete syntax using a homomorphism that does not glue more than abstract-syntax semantics. Since in a user manual concrete syntax should be described by an equational grammar, we should raise a question, whether for any algebra of concrete syntax a corresponding grammar exists. To investigate this problem, we need the concepts of a *skeleton function*.

A function  $f$  on words over an alphabet  $A$  is said to be a *skeleton function* if there exists a tuple of words  $(w_1, \dots, w_k, w_{k+1})$  over  $A$ , called *the skeleton of this function* such that

$$f.(x_1, \dots, x_k) = w_1 x_1 \dots w_k x_n w_{k+1}$$

An example of a skeleton function may be

$$f.(exp-b, ins_1, ins_2) = \text{if } exp-b \text{ then } ins_1 \text{ else } ins_2 \text{ fi}$$

The skeleton of this function is **(if, then, else, fi)**. Notice that the function

$$f.(exp-b, ins_1, ins_2) = \text{if } exp-b \text{ then } ins_2 \text{ else } ins_1 \text{ fi}$$

is not a skeleton function since the order of arguments on the left-hand side of our equation does not coincide with the order on its right-hand side.

In particular cases, a skeleton function may have more than one skeleton. E.g. the one-argument function

$$f : \{a\}^* \mapsto \{a\}^*$$

defined by equation

$$f.(x) = x a$$

has two skeletons  $((), a)$  and  $(a, ())$ , since it may be equivalently defined by the equation

$$f.(x) = a x$$

However, if we change the type of the function  $f$  to  $f : \{a, b\}^* \mapsto \{a, b\}^*$  without changing the defining equation, then this function has only one skeleton  $((), a)$ .

A many-sorted algebra will be called a *syntactic algebra* if it is a reachable algebra of words.

A syntactic algebra will be called a *context-free algebra* if all its constructors are skeleton functions. Of course, algebras of abstract syntax are context-free. As was shown in Sec. 2.13, for each such algebra, we can build an equational grammar that defines its carriers and constructors. Similarly, we may assign an equational grammar for any context-free algebra.

**Theorem 2.15-1** For every context-free algebra, there is an equational grammar that generates its carriers. ■

The following theorem is also true:

**Theorem 2.15-2** For every equational grammar there is a context-free algebra with carriers defined by that grammar. ■

**Proof** Let

$$X_1 = \text{pol}_1.(X_1, \dots, X_n)$$

...

$$X_n = \text{pol}_1.(X_1, \dots, X_n)$$

be an equational grammar with the (unique) solution  $(L_1, \dots, L_n)$ . Assume that the polynomials of that grammar are expressed as unions of monomials. The corresponding algebra

**Alg** = (Sig, Car, Fun, car, fun),

is defined in the following way:

- Sig = (Nc, Nf, ar, so)
- Nc = {cn<sub>1</sub>, ..., cn<sub>n</sub>} — carriers' names are arbitrary, but the number of these names must be equal to the number of equations in the grammar,
- Nf = {fn<sub>1</sub>, ..., fn<sub>m</sub>} — function names are arbitrary, but the number of these names must be equal to the number of monomial occurrences in the grammar,
- ar and so are defined in that way, that they correspond to the arities and sorts of monomials in the grammar,
- Car = {L<sub>1</sub>, ..., L<sub>n</sub>},
- Fun — the set of all monomials in our grammar,
- car.cn<sub>i</sub> = L<sub>i</sub> for i = 1, ..., n

Notice now that every monomial in our grammar is (from the definition) a Chomsky's monomial (see Sec. 2.6), hence satisfies the equation:

$$\text{car.cn}_i(x_1, \dots, x_n) = \{s_1\} x_1 \dots \{s_k\} x_k \{s_{k+1}\}$$

This completes the definition of our algebra. Observe that the defined algebra is unique up to the names of carriers and constructors.

We can show that the carriers of **Alg** are closed wrt all its constructors and that the algebra is reachable. For proof see [32]. ■

Below is a simple example showing how to construct an algebra from a grammar. Consider the following grammar of a two-sorted language

Number = 1 | x | Number + Number

Boolean = Number < Number | Boolean & Boolean

For simplicity, curly brackets for function names have been dropped. The operations of our grammar are defined by the following equations (the symbols of concatenation © has been omitted as well) where n-exp and b-exp with indexes denote numerical and boolean expressions, respectively:

$$\text{one.}() = 1$$

$$\text{variable.}() = x$$

$$\text{plus.}(n\text{-exp}_1, n\text{-exp}_2) = n\text{-exp}_1 + n\text{-exp}_2$$

$$\text{less.}(n\text{-exp}_1, n\text{-exp}_2) = n\text{-exp}_1 < n\text{-exp}_2$$

$$\text{and.}(b\text{-exp}_1, b\text{-exp}_2) = b\text{-exp}_1 \& b\text{-exp}_2$$

An equational grammar is said to be *unambiguous* (resp. *ambiguous*) if the corresponding algebra is unambiguous (resp. ambiguous). Intuitively a grammar is ambiguous if there exists a word w that can be generated by that grammars in two different ways<sup>16</sup>. These “different ways” are different elements of the abstract syntax that are coimages of w wrt the abstract-syntax semantics (see Sec. 2.13). For instance, the word 1+1+1 may be generated in two different ways:

plus(1,plus(1,1))

---

<sup>16</sup> The usability of ambiguous grammars also from the perspective of parsing was investigated in 1972 by A.V. Aho and J.D. Ullman in [3].

plus(plus(1,1),1)

As has been already mentioned, a concrete syntax of a programming language will be constructed as a homomorphic image of its abstract syntax. Since these syntaxes will be described by equational grammars, it is important to know which homomorphisms of syntactic algebras do not lead out of the class of context-free algebras.

Let us start with an example of a homomorphism that destroys the context-freeness of an algebra. Let **Alg** be a one-sorted algebra with the carrier  $\{a\}^+$  and with two operations:

$$h.() = a$$

$$f.(x) = x a$$

This algebra is of course, context-free. Now consider a similar algebra with a carrier

$$\{a^n b^n c^n \mid n = 1, 2, \dots\}$$

and constructors

$$h.() = abc$$

$$f.(a^n b^n c^n) = a^{n+1} b^{n+1} c^{n+1}$$

This algebra is not context-free since its carrier is a well-known example of a not context-free language (see [55]), but it is isomorphic with our former algebra where the corresponding isomorphism is:

$$l.a^n = a^n b^n c^n \quad \text{for every } n \geq 1$$

As is easy to see this isomorphism is not a skeleton function.

A homomorphism  $H$  between two syntactic algebras is called a *skeleton homomorphism* (we recall that since syntactic algebra are reachable, such a homomorphism, if exists, is unique (Theorem 2.13-3)) if for every constructor  $fun.fn$  of the source algebra, for which

$$so.fn = cn$$

$$ar.fn = (cn_1, \dots, cn_n)$$

there exists a skeleton  $(s_1, \dots, s_{n+1})$ , such that

$$H.fn.(fun_1.fn.(x_1, \dots, x_n)) = s_1 x_1 \dots s_n x_n s_{n+1}$$

In other words, a homomorphic image of every constructor of the source algebra is a skeleton constructor in the target algebra.

**Theorem 2.15-3** For every syntactic algebra **Alg** the following facts are equivalent:

- (1) **Alg** is context-free,
- (2) every homomorphism into **Alg** is a skeleton homomorphism,
- (3) there exists a skeleton homomorphism into **Alg**.

For proof, see [19].

Let us consider now a simple example of a process of constructing a syntactic algebra for a given algebra<sup>17</sup>. Let the latter be a one-sorted algebra of numbers with three operations:

$$\begin{array}{lll} \text{create-nu.1} & : & \mapsto \text{Number} \\ \text{plus} & : \text{Number} \times \text{Number} & \mapsto \text{Number} \\ \text{times} & : \text{Number} \times \text{Number} & \mapsto \text{Number} \end{array}$$

The corresponding abstract syntax, denote it by **Syn-0**, is defined by the following grammar with only one equation, where **Exp** denotes a language of numerical expressions with constant values:

$$\text{Exp} = \text{create-nu.1.()} \mid \text{plus}(\text{Exp}, \text{Exp}) \mid \text{times}(\text{Exp}, \text{Exp})$$

<sup>17</sup> In more general terms such processes will be discussed in Sec. 3.4.

The first step on our way to final syntax consists in:

- replacing `create-nu.1` by `1`,
- replacing `plus` and `times` by `+` and `*`,
- replacing prefix notation by infix notation.

This step corresponds to the following homomorphism:

$$\begin{aligned} H.[\text{create-nu.1}()] &= 1 \\ H.[\text{plus}(\text{exp}_1, \text{exp}_2)] &= (H.[\text{exp}_1] + H.[\text{exp}_2]) \\ H.[\text{times}(\text{exp}_1, \text{exp}_2)] &= (H.[\text{exp}_1] * H.[\text{exp}_2]) \end{aligned}$$

This is of course a skeleton homomorphism and the corresponding context-free grammar is the following:

$$\text{Exp} = 1 \mid (\text{Exp} + \text{Exp}) \mid (\text{Exp} * \text{Exp})$$

In the second and the last step of syntax construction we would like to allow dropping out “unnecessary parentheses”, e.g. writing `1+1+1` instead of `(1+(1+1))` and analogously for multiplication. Unfortunately this turns out to be impossible since each homomorphism which removes parentheses has to satisfy the equations:

$$\begin{aligned} H.[(\text{exp}_1 + \text{exp}_2)] &= H.[\text{exp}_1] + H.[\text{exp}_2] \\ H.[(\text{exp}_1 * \text{exp}_2)] &= H.[\text{exp}_1] * H.[\text{exp}_2] \end{aligned}$$

but this would mean that it glues expressions with different denotations, e.g.

$$H.[(1+1)*(1+1)] = H.[((1+(1*1))+1)] = 1+1*1+1$$

Although  $H$  is a skeleton homomorphism, which implies that its target grammar

$$\text{Exp} = 1 \mid \text{Exp} + \text{Exp} \mid \text{Exp} * \text{Exp}$$

is context-free, the corresponding algebra is more ambiguous than the algebra of integers, hence a denotational semantics of this syntax into the algebra of numbers does not exist.

A known traditional way of solving this problem as e.g. in Algol ([7] and [84]) or in Pascal [62] consists in reconstructing the whole model of the language by introducing to the algebra of denotations and to the algebra of syntax three carriers **Com** (component), **Fac** (factor) and **Exp** (expression) and the following signature:

|                     |                           |                      |                                     |
|---------------------|---------------------------|----------------------|-------------------------------------|
| <code>c-to-e</code> | : <b>Com</b>              | $\mapsto$ <b>Exp</b> | component to expression identically |
| <code>+</code>      | : <b>Exp</b> + <b>Com</b> | $\mapsto$ <b>Exp</b> | addition                            |
| <code>f-to-c</code> | : <b>Fac</b>              | $\mapsto$ <b>Com</b> | factor to component identically     |
| <code>*</code>      | : <b>Fac</b> * <b>Com</b> | $\mapsto$ <b>Com</b> | multiplication                      |
| <code>1</code>      | : <b>Fac</b>              | $\mapsto$ <b>Fac</b> | the generation of 1 as a factor     |
| <code>e-to-c</code> | : <b>Exp</b>              | $\mapsto$ <b>Fac</b> | expression to factor identically    |

The corresponding grammar of abstract syntax is the following:

$$\begin{aligned} \text{Exp} &= \text{c-to-e}(\text{Com}) \mid +(\text{Exp}, \text{Com}) \\ \text{Com} &= \text{f-to-c}(\text{Fac}) \mid *(\text{Fac}, \text{Com}) \\ \text{Fac} &= 1 \mid (\text{Exp}) \end{aligned}$$

and for the first (isomorphic) transformed syntax:

$$\begin{aligned} \text{Exp} &= (\text{Com}) \mid (\text{Exp} + \text{Com}) \\ \text{Com} &= (\text{Fac}) \mid (\text{Fac} * \text{Com}) \\ \text{Fac} &= 1 \mid (\text{Exp}) \end{aligned}$$

In this grammar the names of identity functions have been omitted, which, however, does not destroy the unambiguity of the grammar, since these names appear in the elements of different carriers.

Now we can define a skeleton homomorphism that removes parentheses in each of three sorts of expressions:

$$\begin{aligned} E.[(\text{val})] &= \text{val} \\ E.[(\text{val} + \text{exp})] &= E.[\text{exp}] + S.[\text{val}] \end{aligned}$$

$$\begin{aligned}
C.[(fac)] &= C.[fac] \\
C.[(fac * val)] &= F.[fac] * C.[val] \\
F.[1] &= 1 \\
F.[(exp)] &= (exp)
\end{aligned}$$

This leads to the following context-free grammar

$$\begin{aligned}
Exp &= Com \mid Exp + Com \\
Com &= Fac \mid Fac * Com \\
Fac &= 1 \mid (Exp)
\end{aligned}$$

This grammar may be also written in a direct way in using the constructor of iteration:

$$\begin{aligned}
Exp &= Com [+ Com]^* && \text{an expression is a sum of components} \\
Com &= Fac [* Fac]^* && \text{a component is a multiplication of factors}^{18} \\
Fac &= 1 \mid (Exp) && \text{a factor is a constant or an expression in parentheses}
\end{aligned}$$

Observe that the parentheses-removal homomorphism is not an isomorphism, since it glues  $(1+(1+))$  and  $((1+1)+1)$  into  $1+1+1$  and similarly for multiplication. However it does not glue “to much” since addition and multiplication are associative. On the other hand from expression  $((1+1)*(1+1))$  it removes only external parentheses.

The denotational homomorphism for our grammar is now the following:

$$\begin{aligned}
Se.[val] &= Ss.[val] \\
Se.[exp + val] &= Se.[exp] + Sc.[val] \\
Ss.[fac] &= Sc.[fac] \\
Ss.[fac * val] &= Sc.[fac] * Ss.[val] \\
Sc.[1] &= 1 \\
Sc.[(exp)] &= Se.[exp]
\end{aligned}$$

Notice that the above equations express the school rules of priority of multiplication over addition.

#### Commentary 2.15-1

The reader to whom we have promised that denotational models of programming languages will offer readable definitions may have some doubts in this moment. So far, the simple language of arithmetic expressions that is very well known to every ground-school student has been described in a rather complicated way and moreover using advanced mathematics. This, of course, requires a commentary.

First, what we can say to a student in a simple way, when “talking” to a computer, we have to express in a way appropriate for the interpreter. That “appropriate way” is a denotational homomorphism, which may be mapped one-to-one into a code of an interpreter.

Second, the discussed language serves only to illustrate the denotational method in an elementary example. The real advantage of the method will be appreciated (we hope) when we introduce more advanced programming mechanisms such as declarations, types, instructions, recursive procedures, objects, etc. whose definitions require more advanced mathematical tools.

Third, in writing a user’s manual for our language, we may directly refer to our acquaintance with school mathematics by saying that numerical expressions can be written and are calculated in a “usual way”, which frees us from the necessity of showing a grammar. However, as we shall see in Sec. 3.4 there are better solutions to that problem called *colloquial syntax*.

Two following lessons may be learned from our exercise:

First, the description of the simple operation of dropping out unnecessary parentheses requires rather complicated and not very intuitive grammar. Such a grammar is necessary for the implementor but not for the user, who can be simply informed that numerical expressions are written and understood in a “usual way”.

Second, the idea of dropping parentheses came out only at the level of second syntactic algebra, when the two formers have already been defined. Therefore, to implement the parenthesis-free notation one has to restart the construction of the model from scratch. In our simple example, this does not lead to too much work, but in real

<sup>18</sup> Note the difference between the operation of multiplication  $*$ , e.g. as in  $11 *$  and the operation of the iteration of languages  $^*$ , e.g. as in  $[+ Com]^*$ .



situations, things may look different. To avoid such problems, one should think about syntax as early as on the level of the algebra of denotations. This, however, contradicts the philosophy “from denotations to syntax” and also ruins the principle that denotations should be constructed in a maximally simple way.

The above problems were investigated in [30], [32] and [40]. A solution suggested there consists in assuming that the programmer’s syntax that will be called *colloquial syntax* does not need to be a homomorphic image of concrete syntax. In our example concrete syntax would be defined by the grammar:

$$\text{Exp} = 1 \mid (\text{Exp} + \text{Exp}) \mid (\text{Exp} * \text{Exp})$$

and colloquial syntax — which allows for (although it does not force) the omission of parentheses — would be defined by the grammar:

$$\text{Exp} = 1 \mid (\text{Exp} + \text{Exp}) \mid (\text{Exp} * \text{Exp}) \mid \text{Exp} + \text{Exp} \mid \text{Exp} * \text{Exp}$$

Observe that the algebra of colloquial syntax is not only not-homomorphic to the former but is even not similar since it has a different signature (has more constructors).

Note, however, that it is easy to define a transformation that would map our colloquial syntax “back” into concrete syntax by adding the “missing” parentheses. Such a transformation will be called a *restoring transformation*. In practice, this approach leads to a user manual that contains a formal definition of concrete syntax (a grammar) plus an informal rule which says, e.g., that parentheses may be omitted in the “usual way”<sup>19</sup>.

In the general case, a restoring transformation may be described formally or informally according to the complexity of colloquialization. Its formal definition is, however, always necessary for implementors who have to write a procedure that converts each colloquial program into its concrete version.

More on colloquial syntax in **Lingua** in Sec. 7.4.

In the end, one methodological remark seems necessary. Languages discussed in this section covered only expressions without variables. Such a case has, of course, no practical value, and it was chosen only to make examples of algebras and corresponding grammars possibly simple. Starting from Sec. 3.5 we shall discuss methods of constructing denotational models for more realistic languages.

## 2.16 Abstract-syntax grammar is LL(k)

Our equational grammars are equivalent to (well known in the literature) context-free grammars and the latter play an important role in the theory of the syntax of programming languages. Especially wanted context-free grammars are LL(k) *grammars*, since their corresponding parsers are efficient and simple to build. To show that our abstract syntax grammars are LL(k), let’s redefine this concept for equational grammars.

Consider an arbitrary equational grammar **EG** that generates a tuple of languages (**Lan**-1,...,**Lan**-n) over an alphabet **Ter** of characters called *terminals*. The elements of **Lan**-i’s will be called *words*. Every equation of **EG** is the following formula

$$\text{Syn-}i = w\text{-}i1 \mid \dots \mid w\text{-}ip(i) \quad \text{for} \quad 1 \leq i \leq n \quad (7.2-1)$$

where:

- **Syn**-i are metavariables corresponding to syntactic domains; we shall call them *nonterminals*,
- **w**-ij are metawords written over an alphabet  $\text{Alp} = \text{Ter} \mid \{\text{Syn-}1, \dots, \text{Syn-}n\}$ ,

Our grammar will be said to be *strongly prefixed*, if every **w**-ij is not empty, and starts with a terminal. Let’s define an auxiliary function of the k-the prefix of a word (**a**-1,...,**a**-n):

$$\begin{aligned} \text{prefix} : \text{Alp}^{c*} \times \{1, 2, \dots\} &\mapsto \text{Alp}^{c*} \\ \text{prefix}(w, k) = & \\ w = () &\rightarrow () \\ \text{let} & \end{aligned}$$

<sup>19</sup> As we are going to see in Sec. Sec. 7.2 and 7.3 the situation may be a little more complicated.

$$\begin{array}{ll}
 (a_1, \dots, a_n) = w \\
 n \leq k & \rightarrow w \\
 \text{true} & \rightarrow (a_1, \dots, a_k)
 \end{array}$$

For a positive integer  $k$ , a strongly prefixed grammar with equations (7.2-1) is said to be a  $LL(k)$  grammar<sup>20</sup>, if for every index  $1 \leq i \leq n$ , any two different metawords in the  $i$ -th equation,  $w_{ij}$  and  $w_{ip}$ , have different  $k$ -th prefixes, i.e.,  $\text{prefix}(w_{ij}, k) \neq \text{prefix}(w_{ip}, k)$ . Note that metawords of different equations do not need to satisfy this condition.

In a  $LL(k)$  grammar, given a word  $w$  to be parsed, and a non-terminal  $Syn_i$  that determines the category of this word, we need to look ahead not more than  $k$  first characters of  $w$  to identify the grammatical clause to be used in parsing  $w$ . This property of  $LL(k)$  grammars allows to build for them relatively simple deterministic parsers.

As is easy to check, our abstract-syntax grammar is  $LL(k)$  for some  $k$ , since all green prefixes of clauses are different to each other.

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<sup>20</sup> The original concept of a  $LL(k)$  grammar is not restricted to strictly prefixed grammar, but in that case the definition is a little more complicated, and requires the introduction of some additional concepts. On the other hand, the restriction to strictly prefixed grammar is not harmful for our model, since our abstract-syntax and concrete-syntax grammars will be strictly prefixed anyway.



## 3 AN INTUITIVE INTRODUCTION TO DENOTATIONAL MODELS

### 3.1 How did it happen?

Mathematicians building mathematical models of programming languages were usually assuming (as in mathematical logic) that a programming language should be described by three mathematical entities:

1. **Den** — *denotations*, which in our model constitute a many-sorted algebra (Sec. 2.12),
2. **Syn** — *syntax*, which in our model is an algebra similar to the former (has the same signature),
3. **Sem : Syn  $\mapsto$  Den** — *semantics*, that associates denotations to syntactic elements, and in our model is a homomorphism between two mentioned algebras.

Intuitively speaking, a denotational semantics describes the meaning of every complex syntactic object as a composition of the meanings of its components. This property of semantics — called *compositionality* — allows for the description of complex objects by means of so-called *structural induction*.

It should be mentioned in this place that denotational (compositional) models of semantics — which for mathematicians have always been an obvious choice — have not been used in the first formal models of programming languages. Similarly to the prototypes of sewing machines that were mechanical arms repeated the movements of a tailor, and to the first steamboat engine driving oars, the early formal definitions of programming languages were mathematical descriptions of virtual computers executing programs<sup>21</sup>.

Such model of semantics, called later *operational semantics*, were developed by several authors (see e.g., G. Plotkin [85], [86] and C. Jones [63]).

However, the road to denotational semantics wasn't simple either. As was already mentioned, early denotational models of programming languages were characterized by great mathematical complexity. Technically it was the consequence of the assumption that two following mechanisms were indispensable in high-level programming languages:

1. the jump instruction **goto** that transfers program execution from one line of code to another one; this mechanism was available in practically all programming languages in the years 1960/70, and was inherited from low-level languages, where it was the only tool for building logical structures of programs,
2. procedures that may take themselves as parameters; this construction was present in Algol 60 (see [7]) considered by academic community of 1960. as an indisputable standard.

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<sup>21</sup> First metalanguage used to write such semantics in the 1970. was developed in IBM laboratory Vienna and was called Vienna Definition Language (VDL). Later some members of the IBM team have created a lab on the Danish Technical University in Lyngby with the aim of writing a denotational semantics in a metalanguage called Vienna Development Method (VDM) [15]. This language was used, among other applications, to describe the semantics of two programming languages — Ada and Chill. In the case of the former, that was expected to become a universal programming language of all times, the process of writing its semantics resulted in repairing many inaccuracies of the language, and in developing first Ada compiler. Unfortunately, both Chill and Ada were excessively complex, and hence have never become commonly used.

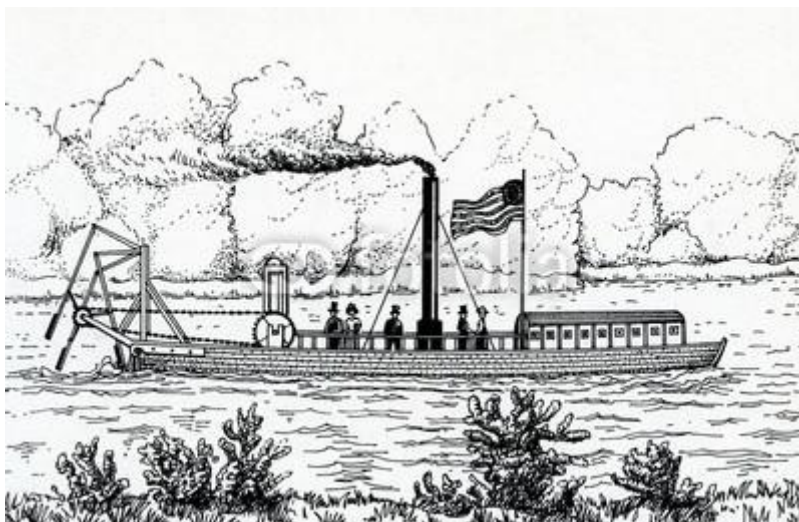


Fig. 3.1-1 Steamboat moving oars

The requirement of having **goto**'s has led to a technically rather complex model of continuations<sup>22</sup>. That semantics was not only technically complex but above all quite far from programmers' intuition. Independently, at the turn of the 1960-ties to 1970-ties, IT professionals began to be aware of a risk imposed by **goto** instruction (see [49]). Programs with **goto**'s were difficult to understand, and therefore not always behave as expected. As a consequence **goto**'s were abandoned in favor of structural programming mechanisms such as **if-the-else**, **while-do-od** and similar.

The continuation model, although technically complex, was based on a traditional mathematics. This can't be said about the model of procedures that take themselves as parameters. Notice that in this case we do not talk about recursive procedures that call themselves in their bodies — such a mechanism is described in this book by fixed-point equations — but about constructions of the type  $f.f$ , where a function takes itself as an argument. Such functions were not known to mathematicians, because they can't be described on the ground of classical set theory, let alone that mathematicians never needed such functions.

In Algol 60 the construction  $f.f$  was implemented in such a way, that a procedure  $f$  was receiving as a parameter not exactly itself, but a copy of its own code, which was inserted into its body during compilation. Such an operation was called *copy rule*. Mathematicians of the decade of 1960. were fascinated by this construction because it was challenging the existing concept of a function. As a consequence, the theory of *reflexive domains* was created by Dana Scott and Christopher Strachey [89] and was later described in detail by J.E. Stoy in a monograph [88]<sup>23</sup>. Although some mathematicians were investigating reflexive domains, for software engineers this theory was even more difficult, and less intuitive than continuations. Pretty soon it turned out also that the self-applicability of procedures was even more error-prone than the use of **goto**'s. Consequently, in later programming languages, self-applicable procedures were abandoned. Unfortunately, some researchers decided that denotational semantics should be abandoned as well.

In the denotational model discussed in this book we use neither continuations nor reflexive domains. In our model the denotations of instructions are state-to-state functions where a state "carries" everything that a program needs to be executed: data, types, procedures, classes etc. Simplifying a little a state is a function that maps identifiers into these mathematical items. The concept of a state is a natural generalization of a concept of a

<sup>22</sup> First author who introduced that concept — although under a different name of *tail functions* — was Antoni Mazurkiewicz [75] in 1971. Under the name of continuations it was introduced in [90] in 1974.

<sup>23</sup> To our colleagues mathematicians we may explain that the idea of reflexive domains was in fact a "hidden realization" of copy rule. The authors of this model used the fact that functions definable by programs are computable, hence can be "numbered" with natural numbers — each function  $f$  may be given a unique number  $n(f)$ . In this model  $f(f)$  meant  $f(n(f))$  which can be modelled on the ground of classical set theory. That was in fact a mathematical application of copy rule since  $n(f)$  may be regarded as the code of procedure  $f$ .

valuation known by mathematicians since the pioneering works of Alfred Tarski [91]. Tarski defined the meanings of expressions as functions mapping valuations of variables

$$\text{val} : \text{Valuation} = \{x, y, z\} \rightarrow \text{Value}$$

into values. E.g., the meaning of an expression

$$2x+4y$$

was a function

$$F.[2x+4y] : \text{Valuation} \rightarrow \text{Number}$$

such that

$$F.[2x+4y].\text{val} = 2*\text{val}.x + 4*\text{val}.y$$

From there only one step to an observation that the meaning of an instruction

$$x := 2x + 4y$$

is such a transformation of valuations where the value of  $x$  in the new valuation is the value of the expression  $2x+4y$  in the former. This idea was applied in [18], published in 1971, where Andrzej Blikle described a prototype of a denotational semantics of a very simple programming language.

In turn, the inspiration to abandon the model of reflexive domains came to me from the book of Michael Gordon [59], where the author treats Scott's reflexive domains as "usual sets" with the following commentary on page 29:

*We shall not discuss the mathematics involved in Scott's theory at all; our approach to recursive equations<sup>24</sup> is similar to an engineering approach to differential equations, namely we assume they have solutions but don't bother with the mathematical justification.*

Andrzej Blikle read Gordon's book in the year 1981 during a train ride from Copenhagen to Århus, where he was going to meet Peter Mosses, a strong proponent of the theory of Dana Scott. The book was, for him, a significant breakthrough since, for the first time, he was reading a semantics of a programming language with an understanding not only of its mathematics but also of its IT content. The treatment of reflexive domains as "usual sets" was a real simplification. He also had the impression that this informal treatment did not lead to any mathematical problems. Only later, he realized that Gordon was actually not dealing with self-applicable functions.

The approach of Michael Gordon, although intuitively simple, was mathematically not entirely acceptable since reflexive domains are not "usual" sets. It wasn't, therefore, clear, whether his model did not include inconsistencies.

To cope with this problem, A.Blikle and A. Tarlecki published in 1983 a paper [39], in which they constructed a denotational model of a programming language, where the domains of denotations are sets, and the denotations of instructions are state-to-state transformations. This approach stimulated in 1980-ties the creation of a metalanguage **MetaSoft** [29] in the Institute of Computer Science of the Polish Academy of Sciences. And this is the approach that we shall discuss and further developed in this book.

## 3.2 From denotations to syntax

All early works on the semantics of programming languages were devoted to building semantics for existing languages. This fact has led to a tacit assumption that in designing a language, the syntax should come first into the play. Of course, there is a certain logic in this way of thinking, since how can we build a model for something that does not yet exist? After all, astronomers were describing the mechanics of celestial bodies when the Sun and the planet were already there.

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<sup>24</sup> M. Gordon is talking here about recursive domain-equations, which, in some case of non-continuous domain operators, lead to D. Scott's reflexive domains.

This way of thinking has, however, a particular vulnerability, since computer science cannot be compared to astronomy, physics, or biology, where we describe the world around us. Building a programming language is an engineering task, such as constructing a bridge or an airplane. Would any engineer ever think of first constructing a bridge basing on common sense and only then making all necessary calculations? Such a bridge would certainly collapse.

In our approach, we reverse the traditional order where one first builds a syntax, and only later defines its meaning. We will build a language starting from an algebra of denotation from which syntax will be derived in such a way that a denotational semantics exists. This construction was sketched in Sec. 2.13.

An experimental programming language developed in this book is called **Lingua**. This Italian name has been suggested by Andrzej Blikle to commemorate the circumstances under which — working as a scholar of Italian government from October to December 1969 — he wrote his habilitation thesis later published in *Dissertationes Mathematicae* [18]. During three months in the Istituto di Elaborazione dell'Informazione in Pisa he described a denotational semantics of a very simple programming language, although he didn't call his semantics in this way. The name “denotational semantics” was used for the first time in a joint work by D. Scott and Ch. Strachey [89]. Only eighteen years later, in the year 1987, Andrzej Blikle described (in [30]) the idea of deriving syntax from denotations.

### 3.3 Why we need denotational models of programming languages?

A denotational model of a programming language serves as a starting point for the realization of three tasks:

1. building an implementation of the language, i.e., its interpreter or compiler,
2. creating rules of building correct specified programs in this language,
3. writing a user manual.

When designing our language in this book, we shall observe two fundamental (although not quite formal) principles:

#### ***First Principle of Simplicity***

*A programming language should be as simple to understand and easy to use as possible without harming its functionality, mathematical clarity, and completeness of its description.*

#### ***Second Principle of Simplicity***

*The same applies to the manual of the language and to the rules of building correct programs.*

These principles shall be fulfilled by:

1. making the syntax of the language as close as possible to the language of “usual” mathematics, e.g., whenever it is common, we allow infix notation and the omission of “unnecessary” parentheses,
2. making the semantics of the language easy to understand by the user rather than convenient for the implementor; for the latter, an equivalent implementation-oriented model may be written.
3. making the structure of the language (i.e., program constructors) leading to possibly simple rules of constructing correct programs (Sec. 8 and Sec. 9),

Particular attention should be given to point 3. because the simplicity of the rules of building correct programs leads to a better understanding of programs by programmers. This fact was realized already in the decade of 1970. and has led to the elimination of **goto** instructions. This decision led to a significant simplification of program structures, which increased their reliability. On the other hand, it did not limit the functionality of programming languages.

Following point 3, we will sometimes — as typical in mathematics — "forget" about the difference between syntax and denotations. E.g., we will talk about the value of an arithmetic expression  $x + y$ , rather than about the value generated by its denotation. We will say that the instruction  $x:=y+1$  modifies the value of  $x$ , instead of saying that the denotation of this instruction modifies a memory state at variable  $x$ , etc. Of course, at a formal level, we shall precisely distinguish syntax from denotations.

### 3.4 Five steps to a denotational model

Building up **Lingua**, we refer to an algebraic model described in Sec. 2.11 to Sec. 2.16. It corresponds to the diagram of three algebras shown in Fig. 3.4-1. We build it in such a way that the equation:

$$A2D = A2C \bullet C2D$$

is satisfied, which guarantees the existence of a denotational semantics of our language.

The construction of a denotational model begins with a description of an algebra of denotation **AlgDen**. Then from the signature of **AlgDen** we derive an *algebra of abstract syntax* **AlgAbsSyn**, and, precisely speaking a context-free grammar that describes this algebra. The first of these steps is creative since it comprises all the significant decisions about a future language. In turn, the second step can be performed algorithmically.

Since abstract syntax is usually not convenient for programmers, we build an *algebra of concrete syntax* **AlgConSyn**. In typical situations, we do it by replacing prefix notation by infix notation and introducing more intuitive names of constructors. In our approach the corresponding abstract-to-concrete homomorphism **A2C** will be an adequate homomorphism, which guarantees the existence of a unique homomorphism:

$$C2D : \mathbf{AlgConSyn} \mapsto \mathbf{AlgDen}$$

(*concrete semantics*), which is the semantics of concrete syntax. In this way, we create the main components of our denotational model.

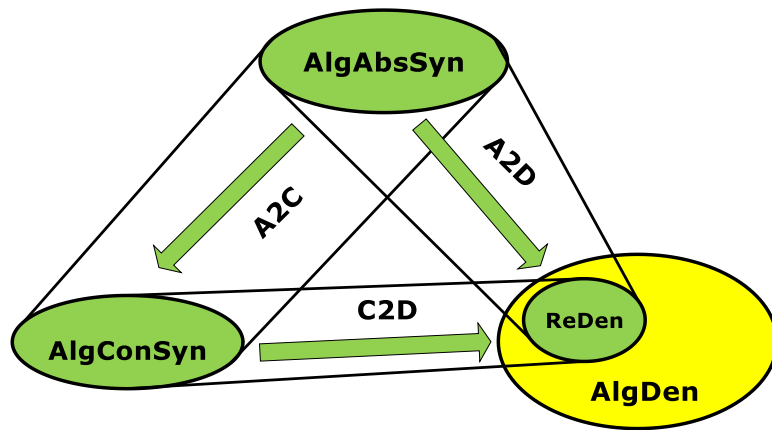


Fig. 3.4-1 Basic algebraic model of a programming language

The step from abstract syntax to concrete syntax is creative — although rather simple. For instance, instead of writing  $+(a, b)$  we write  $(a + b)$  and instead of writing

$\text{if}(\text{greater}.(x, 0), \text{assign}.(x, \text{plus}.(x, 1)), \text{assign}.(x, \text{minus}.(x, 1)))$

we write

$\text{if } x > 0 \text{ then } x := x + 1 \text{ else } x := x - 1 \text{ fi}$

The next step in building a user-friendly syntax consist in introducing so called *colloquialisms*. For instance instead of writing

$(a + (b + (c * d)))$

we shall write

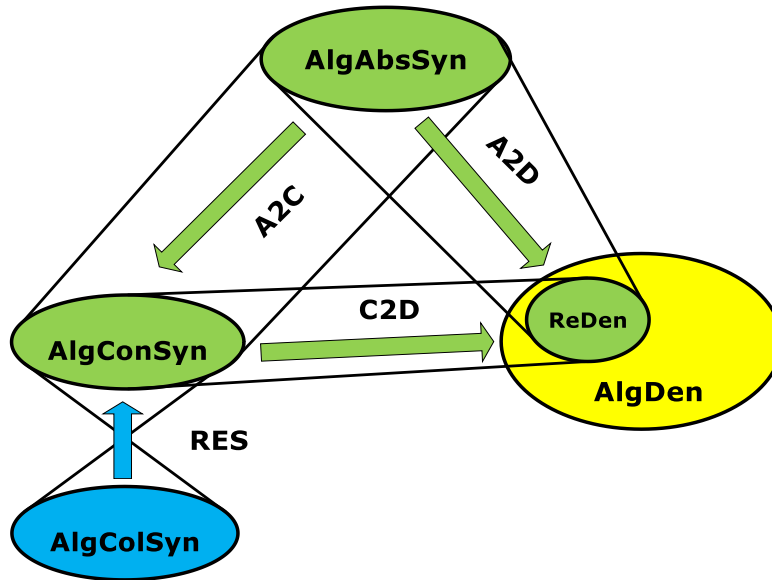


$a + b + c * d$

assuming that multiplication binds stronger than addition, and that “the remaining” parentheses are added from left to right. The introduction of colloquialisms into concrete syntax leads to an *algebra of colloquial syntax* **ColSyn** (Fig. 3.4-2), which most frequently has a different signature than concrete syntax, and therefore can’t be a homomorphic image of it. However, we make sur that there exists an implementable *restoring transformation*

$$\text{RES} : \text{AlgColSyn} \mapsto \text{AlgConSyn}$$

that transforms colloquial syntax back to the concrete one, e.g., by adding the missing parentheses.



**Fig. 3.4-2 An algebraic model of a language with colloquial syntax**

In a programmer’s manual, a language with colloquialisms is described by a grammar of concrete syntax with additional clauses and a restoring transformation (Sec. 7.4). For instance, we explain that in writing arithmetic expressions, we can skip parentheses while maintaining the priority of multiplication and division over addition and subtraction.

In such a case, an implementor receives a standard denotational model of a language plus a formal definition (algorithm) of restoring transformation. The execution of a program consists then of two steps:

1. a pre-treatment of the source code by a restoring transformation,
2. an interpretation or compilation of concrete-syntax code.

Summing up our considerations, the construction of a denotational model of a programming language correct-program constructors proceeds in five steps:

1. In the first step, we build an algebra of detonations **AlgDen** that includes the denotations of the future syntax as well as their constructors. In that step, significant decisions are taken about the functionality of the language. A language designer must specify the repertoire of constructors of **AlgDen** in such a way that the corresponding (unique) reachable subalgebra contains all the elements that we want to access through syntax. This will be illustrated and explained in Sec. 6. In the earlier Sec. 3.5 and Sec. 5 we build technical fundamentals for the algebra of denotations — data- and type-oriented algebras, objects, classes and states.
2. The signature of algebra **AlgDen** uniquely determines the algebra of abstract syntax **AlgAbsSyn** and the corresponding homomorphism (abstract semantics) **A2D**. Formally this step (Sec. 7.2) leads from the signature of **AlgDen** to an equational grammar of **AlgAbsSyn**, and can be performed algorithmically.
3. Since abstract syntax is not user-friendly, we transform it (Sec. 7.3) in a homomorphic way to a concrete syntax **AlgConSyn**, which is closer to programmers’ syntax. We make sure that this homomorphism is

adequate which guarantees the existence a denotational semantics (a homomorphism  $C2D : \mathbf{AlgConSyn} \mapsto \mathbf{Den}$ ).

4. In the fourth step, we introduce colloquialisms (Sec. 7.4) — which make our language even more user-friendly — and describe the restoring transformation. This step is creative again. The grammar of colloquial syntax emerges from the grammar of concrete syntax by adding to it some new grammatical clauses.
5. In the last step we build tools for the construction of correct programs (Sec. 9). In our opinion this step should be regarded as an inherent phase in designing a programming language. It should be the responsibility of a language designer to choose such programming mechanisms which make the corresponding constructors of correct programs sufficiently easy to use.

In the end, let us emphasize the following:

*Lingua is not regarded as a prototype of a standalone programming language,  
but only as an example  
illustrating what a denotational model of a programming language may look like.*

Consequently, the lack of specific, concrete constructions in **Lingua** does not mean that we regard them as unimportant; rather, it is an attempt to keep our book within a reasonable size (it is pretty long, though).

### 3.5 Six steps to the algebra of denotations

On the ground of our model we suggest a certain systematic way of getting to an algebra of denotations of a future object-oriented language.

1. In the first step we decide about the categories of data that we want to have in the language and a corresponding set of constructors. Typically, we may start by defining some simple data, e.g., numbers or texts, and some structured data, e.g., lists or arrays.
2. The mentioned categories of data correspond to datatypes. Datatypes correspond to sets of data, but they are not such sets. They are independent mathematical beings that only describe such sets. This solution allows us to assume that whenever we build a (new) data, we “simultaneously” build its type.
3. To formalize the described mechanism we introduce *typed data* that are pairs consisting of data and their types. Each constructor of typed data, given a tuple of arguments  $((\text{dat-1}, \text{typ-1}), \dots, (\text{dat-n}, \text{typ-n}))$  builds a new typed data by applying a data constructor to  $(\text{dat-1}, \dots, \text{dat-n})$  and a corresponding type constructor to  $(\text{typ-1}, \dots, \text{typ-n})$ .
4. Typed data constitute one of two categories of *values*. The second category are *objects* that are pairs consisting of an *objecton* and its type. Objectons are typed memory structures and their types are the names (identifiers) of corresponding classes. Classes, in turn, are structure that carry types, methods and objectons.
5. Since denotations in our model are (with some exceptions) functions on *states*, in the last but one step we define states. The latter carry classes, values assigned to variables (identifiers) via references and some other elements of a technical character. References represent typed memory addresses.
6. In the last step we define an *algebra of denotations*, i.e., its carriers and constructors. We will have two major categories of denotations: *applicative denotations* — the denotations of expressions, and *imperative denotations* — the denotations of declarations and instructions.

Although in steps 1., 2. and 3. we might talk about building algebras, we do not formalize this fact, since in building our model we refer merely to their elements and constructors. In turn, in the case of denotations, we define a corresponding algebra explicitly, since later on we shall derive from this algebra our algebras of syntaxes.

### 3.6 **Lingua** as a strongly-typed language

In a manual of SQL ([52] p. 786), we can read the following sentence<sup>25</sup>:

“If we do not provide (...) correct values to functions as their arguments, we should not expect consistent results.”

Contrary to this philosophy, **Lingua** will be constructed in such a way that whenever a program provides unexpected values to a function, this function will generate an error message and/or initiate a recovery action. To achieve this goal, we equip **Lingua** with a *typing discipline* partly announced in Sec. 3.5. In **Lingua** “incorrect values” means “values of not acceptable types”. Types will be used in the descriptions of the following mechanisms:

1. the declarations of variables,
2. the declarations of user-defined types,
3. the evaluation of expressions,
4. the execution of assignment instructions,
5. passing arguments to operations on values,
6. passing actual parameters to all three categories of procedures — imperative, functional and object constructors,
7. returning reference parameters at the end of imperative-procedure calls,
8. returning values of functional procedures,
9. defining the types of formal parameters of all categories of procedures.

---

<sup>25</sup> Andrzej Blikle’s translations from a Polish edition [52].



## 4 DATA, TYPES, VALUES

### 4.1 Data

The first step of designing a programming language in our framework consists in defining *data* and their constructors, i.e., an *algebra of data*. It is to be emphasized in this place that in our model we will have two categories of algebras:

- *first-class algebras* — algebras of denotations and the corresponding algebras of syntaxes,
- *second-class algebras* — algebras of data, datatypes and typed data.

In the first case we make sure that the algebras of denotations have nonempty reachable subalgebras, since that is necessary for the corresponding algebras of syntaxes (reachable by definition), to be not empty. We give names to these algebras such as **AlgDen** or **AlgAbsSyn** and we define homomorphism between them, e.g.:

$$\text{A2D} : \mathbf{AlgAbsSyn} \mapsto \mathbf{AlgDen}$$

The situation with second-class algebras is different. We do not introduce specific metasymbols for them, and we do not care about their reachable parts. However, we still refer to them as “algebras” to emphasize that they represent algebraic structures: a family of sets of elements operations on them.

Passing to our first algebra we recall and reemphasize that in our book we are not building a real programming language, but only indicate how such a language might be designed. Consequently, our operations on data do not need to constitute a practically complete set of operations. They only offer some typical examples of such operations and their definitions.

To begin with, we assume to be given some *simple data* offered by an implementation platform (IP). We shall not define them explicitly assuming that they are just parameters of our model. Let’s assume, therefore, that we have the following (somehow defined) domains of simple data:

```
int   : Integer   = ...
rea   : Real      = ...
boo   : Boolean   = {tt, ff}
tex   : Text      = ...
```

and that all these domains (except **Boolean**) are restricted by limiting the sizes of their elements, e.g.,

$$\text{int} : \mathbf{Integer} = [-2^{31}, 2^{31} - 1]$$

We assume further to have a set of corresponding *IP-constructors*, defined on simple data and again offered by an implementation platform, such as, e.g.,

|  |                                       |
|--|---------------------------------------|
| IP-divide-in : Integer x Integer → Integer | IP-division of integers <sup>26</sup> |
| IP-divide-re : Real x Real → Real          | IP-division of reals                  |

In the general case we may assume that these functions are partial by which we mean that their executions may either yield no value (e.g., looping indefinitely), or return an “unwanted” value<sup>27</sup>. Let

$$\text{dat} : \mathbf{SimData} = \mathbf{Boolean} \mid \mathbf{Integer} \mid \mathbf{Real} \mid \mathbf{Text}$$

<sup>26</sup> We assume that the result of this operation within the range of the IP-integers is the integer part of the rational result of the “mathematical” division of integers.

<sup>27</sup> This may happen, e.g., if the implementation platform provides an addition modulo say  $10^9$ , where  $999999999 + 1 = 0$ .

be the domain of simple data, and let

$\text{ide} : \text{Identifier} = \dots$

be a set of (somehow defined) identifiers. On this ground we define a domain of *data* that are either simple or *structured*:

$\text{dat} : \text{Data} = \text{SimData} \mid \text{List} \mid \text{Array} \mid \text{Record}$   
 $\text{lis} : \text{List} = \text{Data}^{\text{c}*}$   
 $\text{arr} : \text{Array} = \text{Integer} \Rightarrow \text{Data} \mid \{\Omega\}$   
 $\text{rec} : \text{Record} = \text{Identifier} \Rightarrow \text{Data}$

A list is a finite, possibly empty, tuple of arbitrary data. Arrays and records are mappings, i.e., finite functions. Arrays are one-dimensional, but since their elements can be arrays themselves, our model includes arrays of arbitrary (finite) dimensions. The special element  $\Omega$  will be called an *empty data* and will represent a “place in an array” where a data may be inserted. We shall need it to describe the declaration of an array variable.

Identifiers in the domain of a record will be called the *record attributes*.

All domains of data, except **SimData**, will be referred to as *data sorts*, e.g., *integer sort*, *text sort*, *array sort*, etc. At this stage, lists and arrays are not-homogeneous, i.e., may include elements of different sorts, and may be arbitrarily large. Later the constructors of values, i.e. typed data (Sec. 4.3) will assure that all data generated by programs will have “appropriate” structures and sizes. The technique of defining “oversized” domains whose implementable parts are later appropriately “truncated” is typical for denotational models since it leads to simple domain equations. We will frequently use it in the sequel<sup>28</sup>.

To define our data constructors we assume to be given a universal domain **Error** of all “future” error messages and that with every domain of data we associate a corresponding domain with errors, e.g.,

$\text{int} : \text{IntegerE} = \text{Integer} \mid \text{Error}.$

Having defined data domains, we may proceed to the definitions of data constructors. We start with their signatures and give some of their definitions a little later. Since we regard the domains and the constructors of data as parameters of our model, their definitions should be regarded as examples. The names of data constructors are prefixed with **da-** which stands for “data”. Later we will have type constructors, value constructors, denotation constructors etc.

### Comparison constructors

|                      |  |                           |                    |
|----------------------|--|---------------------------|--------------------|
| $\text{da-equal-in}$ | $: \text{IntegerE} \times \text{IntegerE}$ | $\mapsto \text{BooleanE}$ | integer comparison |
| $\text{da-less-in}$  | $: \text{IntegerE} \times \text{IntegerE}$ | $\mapsto \text{BooleanE}$ | integer comparison |
| $\text{da-equal-re}$ | $: \text{RealE} \times \text{RealE}$       | $\mapsto \text{BooleanE}$ | real comparison    |
| $\text{da-less-re}$  | $: \text{RealE} \times \text{RealE}$       | $\mapsto \text{BooleanE}$ | real comparison    |
| $\text{da-equal-te}$ | $: \text{TextE} \times \text{TextE}$       | $\mapsto \text{BooleanE}$ | text comparison    |

In this place we take engineering decisions which data may be compared and in which way.

At this stage we do not introduce logical connectives **and**, **or** and **not** in the domain **BooleanE**. They will come into play only at the level of value expression denotations in Sec. 6.4.1, and this decision will be explained there.

### Integer number constructors

|                         |  |                           |
|-------------------------|--|---------------------------|
| $\text{da-add-in}$      | $: \text{IntegerE} \times \text{IntegerE}$ | $\mapsto \text{IntegerE}$ |
| $\text{da-subtract-in}$ | $: \text{IntegerE} \times \text{IntegerE}$ | $\mapsto \text{IntegerE}$ |
| $\text{da-multiply-in}$ | $: \text{IntegerE} \times \text{IntegerE}$ | $\mapsto \text{IntegerE}$ |

<sup>28</sup> At the level of the algebra of denotations “implementable” would mean “algebraically reachable” (cf. Sec. 2.13). As we remember, only reachable denotations are representable in syntax. However, at the level of data — and later of types and values — we do not need to care about reachability, since these elements won’t have syntactic counterparts. Indeed, if we write e.g. 17.3 in a program, it is not a syntactic representation of the corresponding number, but of an expression whose fixed value is that number. This will be formalized in Sec. 6.4.1.

da-divide-in : IntegerE x IntegerE  $\mapsto$  IntegerE

### Real number constructors

da-add-re : RealE x RealE  $\mapsto$  RealE  
 da-subtract-re : RealE x RealE  $\mapsto$  RealE  
 da-multiply-re : RealE x RealE  $\mapsto$  RealE  
 da-divide-re : RealE x RealE  $\mapsto$  RealE

### Text constructors

da-glue-te : TextE x TextE  $\mapsto$  TextE

### List constructors

da-empty-li :  $\mapsto$  ListE  
 da-put-to-li : DataE x ListE  $\mapsto$  ListE  
 da-head-li : ListE  $\mapsto$  DataE  
 da-tail-li : ListE  $\mapsto$  ListE

### Array constructors

da-change-in-ar : ArrayE x IntegerE x DataE  $\mapsto$  ArrayE  
 da-get-from-ar : ArrayE x IntegerE  $\mapsto$  DataE

### Record constructors

da-create-rc : Identifier x DataE  $\mapsto$  RecordE  
 da-put-to-rc : DataE x RecordE x Identifier  $\mapsto$  RecordE  
 da-change-in-rc : RecordE x Identifier x DataE  $\mapsto$  RecordE      replace an element of a record  
 da-get-from-rc : RecordE x Identifier  $\mapsto$  DataE

Notice that among our constructors, we do not have constructors of identifiers. We return to them at the level of value-expression denotations in Sec. 6.4.1.

The operations on integers and reals will be defined by referring to the corresponding IP-operations. In this place we must take into account that for some arguments these constructors may return either wrong answers or no answers at all. E.g., we can't divide a number by zero, or can't add two numbers if their sum would be too large for the current implementation. In all such cases our data constructors should not be performed in a "standard way", but instead an error message must be generated. Similar restrictions concern the constructors of structured data. For instance, we may wish to set a limit to arrays' volumes.

To systematically incorporate all these limitations, into our model, with some data constructor we associate an auxiliary function called a *trust test*. E.g. with real division we associate a trust test:

trust-divide-re : RealE x RealE  $\mapsto$  Error | {'OK'}

such that whenever it generates 'OK', the data constructor da-divide-re returns a correct arithmetical result. For instance we may set:

trust-da-divide-re.(rea-1, rea-2) =  
   rea-i : Error  $\rightarrow$  rea-i for i = 1, 2  
   rea-2 = 0  $\rightarrow$  'division by zero not allowed'  
   divide-re.(rea-1, rea-2) > max-rea  $\rightarrow$  'overflow'  
   divide-re.(rea-1, rea-2) < min-rea  $\rightarrow$  'underflow'  
   true  $\rightarrow$  'OK'

where divide-re is a "strict" mathematical division, and max-rea and min-rea denote the maximal/minimal reals acceptable in a current implementation. Of course, the predicates

divide-re.(rea-1, rea-2) > max-rea  
 divide-re.(rea-1, rea-2) < min-rea

must be "somehow" implemented, since divide-re is "implementably not available".

We assume that all our trust tests will be transparent for errors (Sec. 2.9). In this book we shall not define trust tests explicitly assuming that they constitute yet another category of parameters of our model.

Given a trust test for the division of reals, the definition of the corresponding data constructor will be the following:

```
da-divide-re.(rea-1, rea-2) =
  trust-da-divide-re.(rea-1, rea-2) : Error  → trust-divide-re.(rea-1, rea-2)
  true                                     → IP-divide-re.(rea-1, rea-2)
```

Another example of a trust test is associated with a constructor that adds an element to a list:

```
trust-put-to-li : DataE x ListE → Error | {'OK'}
trust-put-to-li.(dat, lis) =
  dat : Error      → dat
  lis : Error      → lis
  size.(push.(dat, lis)) > max-li → 'overflow'
  true             → 'OK'
```

where `push` is a “mathematical” functions defined in Sec. 2.2, `size` is a function that somehow computes the memory size necessary to “fit” the list, and `max-li` is a parameter of our model<sup>29</sup>. The definitions of the remaining trust tests are analogous and we assume them to be parameters of our model.

An example of an array constructor may be the following:

```
da-get-from-ar : ArrayE x IntegerE → DataE
da-get-from-ar.(arr, int) =
  arr : Error  → arr
  int : Error  → int
  arr.int = ?  → 'index beyond range'
  arr.int = Ω  → 'element not initialized'
  true        → arr.int
```

Note that our algebra of data is not reachable, since we have not defined any zero-ary data constructors. As was already mentioned, we shall only care about the reachability of algebras at the level of denotations.

In the end, it is to be emphasized that we use trust tests only at the data level. We “sort them out” from the definitions of data constructors to emphasize that establishing them constitutes an essential step in designing a programming language. In the sequel, practically all other constructors that we define throughout the book will perform some adequacy checks of their arguments, but we shall not define these checks as separate trust tests.

## 4.2 Datatypes — the types of data

Having defined data and their constructors we may proceed to the types of data otherwise called *datatypes*. Datatypes describe “internal structures” of data, and will become components of values. Formally datatypes are defined as words, tuples, mappings or their combinations. The categories of datatypes reflect possible structures of data:

|   |  |              |
|---|--|--------------|
| <code>typ : DatTyp =</code>                         |  |              |
| <code>{'integer', 'real', 'boolean', 'text'}</code> |  | simple types |
| <code>{'L'} x DatTyp</code>                         |  | list types   |
| <code>{'A'} x DatTyp x Integer</code>               |  | array types  |

<sup>29</sup> In this definition, and in the majority of definitions in the sequel, we assume that whenever an error appears in a computation, this computation is aborted and the error is signalized, e.g., displayed on a monitor. In our case if both **dat** and **lis** are errors, then `dat-error` is signalized. An alternative to this solution could be, that we first search for possible errors, and only then perform the expected operation. Since such a solution would significantly lengthen our definitions, and made them less readable, we gave it up. After all **Lingua** is only an example.

$\{ 'R' \} \times (\text{Identifier} \Rightarrow \text{DatTyp})$ 

record types

Types of simple data are just words. Symbols 'L', 'A' and 'R' are called *type initials* and indicate the *sorts* of structural types. E.g. ('A', 'integer', 100) is the type of arrays of integers of length 100, and ('L', ('A', 'real', 100)) is the type of lists whose elements are arrays of reals.

In the case of a list-type ('L', typ) we say that typ is the *inner type* of the list type and similarly for array-types. The elements of the domain

 $\text{tyr} : \text{TypRec} = \text{Identifier} \Rightarrow \text{DatTyp}$ 

are called *type records*, e.g.:

employee-type-rec =

```

['ch-name'      / 'text',
'fa-name'       / 'text',
birth-year      / 'integer',
'award-years'   / ('A', 'integer', 10),
'salary'        / 'integer',
'commission'    / 'integer' ]

```

Type records, i.e., records of types, should not be confused with record types, that are types of records (of data). A record type consist of a record initial 'R' and a type record. Examples of composed datatypes may be:

```

('L', ('R', [name/'text', age/'integer'] ) )           a type of lists of records
('A', ('L', ('R', [name/'text', age/'integer'] ) ), 1000 )  a type of arrays of lists of records

```

The definition of the domains of types anticipates the principle that all elements of a list or of an array must have a common type.

To associate data with datatypes, we define for each type a set of data called the *clan* of this type. Formally, we define a function CLAN that with each type assigns its clan:

 $\text{CLAN} : \text{DatTyp} \mapsto \text{Sub.Data}$ 

This function is defined by structural induction:

```

CLAN.'boolean'      = Boolean
CLAN.'integer'      = Integer
CLAN.'real'         = Real
CLAN.'text'         = Text
CLAN.('L', typ)     = (CLAN.typ)c*
CLAN.('A', typ, int) = {0,...,int}  $\mapsto$  CLAN.typ
CLAN.('R', [ide-1/typ-1,..., ide-n/typ-n]) =
  { [ide-1/dat-1,..., ide-n/dat-n] | dat-i : CLAN.typ-i for i = 1;n }

```

An important fact to be signalized in this place is that the union of the clans of all types does not exhaust the domain Data. In other words, there exist data which have no types. An example may be a non-homogeneous list such as, e.g., (123, 'abc', tt). As we will see in the sequel, typeless data will not “happen” in our programs. By introducing types, we restrict the sets of “generable data”, i.e., data that can be generated by programs.

For technical purposes we introduce an auxiliary function of a *sort of a type*:

 $\text{sort-t} : \text{DatTyp} \mapsto \{ \text{'boolean'}, \text{'integer'}, \text{'real'}, \text{'text'}, \text{'L'}, \text{'A'}, \text{'R'} \}$ 

sort-t.typ =

```

typ = 'boolean'       $\rightarrow$  'boolean'
typ = 'integer'       $\rightarrow$  'integer'
typ = 'real'          $\rightarrow$  'real'
typ = 'text'          $\rightarrow$  'text'
typ : { 'L' }  $\times$  DatTyp  $\rightarrow$  'L'
typ : { 'A' }  $\times$  DatTyp  $\times$  Integer  $\rightarrow$  'A'

```

$$\text{typ} : \{\text{'R'}\} \times (\text{Identifier} \Rightarrow \text{DatTyp}) \rightarrow \text{'R'}$$

To define constructors of datatypes we introduce a domain that includes datatypes and errors.

$$\text{typ} : \text{DatTypE} = \text{DatTyp} \mid \text{Error}$$

Now, for every data constructor **da-co** we define a data-type constructor **ty-co** that builds the type of the data built by **da-co**.

### Comparison constructors

|                 |  |                          |
|-----------------|--|--------------------------|
| <b>ty-equal</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-less</b>  | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |

### Arithmetic constructors for integers

|                       |  |                          |
|-----------------------|--|--------------------------|
| <b>ty-add-in</b>      | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-subtract-in</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-multiply-in</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-divide-in</b>   | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |

### Arithmetic constructors for reals

|                       |  |                          |
|-----------------------|--|--------------------------|
| <b>ty-add-re</b>      | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-subtract-re</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-multiply-re</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-divide-re</b>   | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |

### Text constructors

|                   |  |                          |
|-------------------|--|--------------------------|
| <b>ty-glue-tx</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
|-------------------|--|--------------------------|

### List constructors

|                     |  |                          |
|---------------------|--|--------------------------|
| <b>ty-empty-li</b>  | : $\text{DatTypE}$                       | $\mapsto \text{DatTypE}$ |
| <b>ty-put-to-li</b> | : $\text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-head-li</b>   | : $\text{DatTypE}$                       | $\mapsto \text{DatTypE}$ |
| <b>ty-tail-li</b>   | : $\text{DatTypE}$                       | $\mapsto \text{DatTypE}$ |

### Array constructors

|                        |  |                          |
|------------------------|--|--------------------------|
| <b>ty-create-ar</b>    | : $\text{DatTypE} \times \text{IntegerE}$                      | $\mapsto \text{DatTypE}$ |
| <b>ty-change-in-ar</b> | : $\text{DatTypE} \times \text{DatTypE} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |

### Record constructors

|                        |   |                          |
|------------------------|---|--------------------------|
| <b>ty-create-rc</b>    | : $\text{Identifier} \times \text{DatTypE}$                       | $\mapsto \text{DatTypE}$ |
| <b>ty-put-to-rc</b>    | : $\text{DatTypE} \times \text{DatTypE} \times \text{Identifier}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-change-in-rc</b> | : $\text{DatTypE} \times \text{Identifier} \times \text{DatTypE}$ | $\mapsto \text{DatTypE}$ |
| <b>ty-get-from-rc</b>  | : $\text{DatTypE} \times \text{Identifier}$                       | $\mapsto \text{DatTypE}$ |

Below show a few examples of the definitions of these constructors:

|                                      |   |                |
|--------------------------------------|---|----------------|
| <b>ty-equal.(typ-1, typ-2) =</b>     |   |                |
| <b>typ-i : Error</b>                 | $\rightarrow \text{typ-i}$                                    | for $i = 1, 2$ |
| <b>not comparable.typ-i</b>          | $\rightarrow \text{'not comparable'}$                         | for $i = 1, 2$ |
| <b>typ-1 <math>\neq</math> typ-2</b> | $\rightarrow \text{'types of compared values must coincide'}$ |                |
| <b>true</b>                          | $\rightarrow \text{'boolean'}$                                |                |

Here we have introduced a metapredicate **comparable** to indicate the fact that data of some types may be not comparable. E.g. we may assume that simple data are comparable, but structured data are not. Of course other solutions are possible as well.

|                                      |                            |                |
|--------------------------------------|----------------------------|----------------|
| <b>ty-divide-in.(typ-1, typ-2) =</b> |                            |                |
| <b>typ-i : Error</b>                 | $\rightarrow \text{typ-i}$ | for $i = 1, 2$ |

$\text{typ-i} \neq \text{'integer'} \rightarrow \text{'integer expected'}$  for  $i = 1, 2$   
 $\text{true} \rightarrow \text{'integer'}$

$\text{ty-empty-li.typ} =$   
 $\text{typ : Error} \rightarrow \text{typ}$   
 $\text{true} \rightarrow (\text{'L'}, \text{typ})$

$\text{ty-put-to-li}(\text{typ-e}, \text{typ-l}) =$  cons  $\text{typ-e}$  on list  $\text{typ-l}$   
 $\text{typ-i : Error} \rightarrow \text{typ-i}$  for  $i = e, l$   
 $\text{sort-t.typ-l} \neq \text{'L'} \rightarrow \text{'list expected'}$   
 $\text{let}$   
 $\quad (\text{'L'}, \text{typ}) = \text{typ-l}$   
 $\text{typ-e} \neq \text{typ} \rightarrow \text{'conflict of types'}$   
 $\text{true} \rightarrow \text{typ-l}$

$\text{ty-create-ar}(\text{typ}, \text{int}) =$   
 $\text{typ : Error} \rightarrow \text{typ}$   
 $\text{int : Error} \rightarrow \text{int}$   
 $\text{int} \leq 0 \rightarrow \text{'array size must be positive'}$   
 $\text{int} > \text{max\_a} \rightarrow \text{'too large size of array'}$  here  $\text{max\_a}$  is the maximal array size in the current implementation  
 $\text{true} \rightarrow (\text{'A'}, \text{typ}, \text{int})$

$\text{ty-change-in-ar}(\text{typ-a}, \text{typ-e}, \text{int}) =$  for  $i = a, e$   
 $\text{typ-i : Error} \rightarrow \text{typ}$   
 $\text{int : Error} \rightarrow \text{int}$   
 $\text{sort-t.typ-a} \neq \text{'A'} \rightarrow \text{'array-type expected'}$   
 $\text{let}$   
 $\quad (\text{'A'}, \text{typ}, \text{int-a}) = \text{typ-a}$   
 $\text{int} \notin \{0, \dots, \text{int-a}\} \rightarrow \text{'index beyond range'}$   
 $\text{typ-e} \neq \text{typ} \rightarrow \text{'types not compatible'}$   
 $\text{true} \rightarrow \text{typ-a}$

$\text{ty-create-rc}(\text{ide}, \text{typ}) =$   
 $\text{typ : Error} \rightarrow \text{typ}$   
 $\text{true} \rightarrow (\text{'R'}, [\text{ide}/\text{typ}])$

$\text{ty-put-to-rc}(\text{typ-e}, \text{typ-r}, \text{ide}) =$  put  $\text{typ-e}$  to record  $\text{typ-r}$  on attribute  $\text{ide}$   
 $\text{typ-i : Error} \rightarrow \text{typ-i}$   
 $\text{sort-t.typ-r} \neq \text{'R'} \rightarrow \text{'record expected'}$   
 $\text{typ-r.ide} = ! \rightarrow \text{'attribute already exist'}$   
 $\text{true} \rightarrow (\text{'R'}, \text{typ-r}[\text{ide}/\text{typ-e}])$

$\text{ty-change-in-rc}(\text{typ-r}, \text{ide}, \text{typ-e}) =$  check if new type coincides with the former  
for  $i = r, e$   
 $\text{typ-i : Error} \rightarrow \text{typ-i}$   
 $\text{sort-t.typ-r} \neq \text{'R'} \rightarrow \text{'record expected'}$   
 $\text{let}$   
 $\quad (\text{'R'}, \text{typ-rb}) = \text{typ-r}$  -rt for „record type”  
 $\text{typ-rb.ide} = ? \rightarrow \text{'no such attribute'}$   
 $\text{let}$   
 $\quad \text{typ-at} = \text{typ-rb.ide}$  -at for „attribute type”  
 $\text{typ-e} \neq \text{typ-at} \rightarrow \text{'conflict of types'}$   
 $\text{true} \rightarrow \text{typ-r}$



The last constructor will be used in Sec. 4.3 in the definition of a constructor of typed data that replaces a data assigned to an attribute of a record by another data. Here we anticipate the fact that if we replace a data assigned to a record attribute, the new data must have the same type as the former one.

Type constructors will play a double role in our model:

1. they will be used in evaluating value expressions to build the type of the new value,
2. they will be used in type expressions and type declarations to build user-defined datatypes.

### 4.3 Typed data

*Typed data* are pairs consisting of a data and its type. Their constructors constitute a fundament for future value-expression denotations. The domain of typed data is the following:

$$\text{tyd} : \text{TypDat} = \{(\text{dat}, \text{typ}) \mid \text{dat} : \text{CLAN.typ}\} \quad (4.3-1)$$

In this place we should explain why we decided to operate on typed data, rather than on data alone, despite the fact that if a data has a type then this type is unique type (Sec. 4.2)? There are at least five reasons of our decision:

1. Not all data have types.
2. We want to show explicitly how the modifications of data go “in parallel” with the modification of their types. In this way, we also suggest a specific solution for **Lingua** implementation.
3. Whenever a typed data becomes an argument of an operation, or is to be assigned to a variable or to a formal parameter of a procedure, we have to check the coincidence of the type of this data with an expected type of an argument, a variable or a parameter respectively. In all such cases having an explicit type of a data is just handy.
4. As we will see in Sec. 4.4, typed data will constitute just one category of values, whereas the another category will be constituted by objects consisting of an objecton and its type. In this case one objecton may be associated with many different types.
5. In Sec. 5.4.2 we will introduce a covering relation between types such that wherever a value of **typ1** is expected, we can use a value of **typ2**, provided that **typ1** covers **typ2**.

A typed data that carries a simple data is called *simple typed-data* and analogously are understood *structural typed-data*. The constructors of typed data will “call” the corresponding constructors of data and of types. To describe this mechanism we expand the earlier introduced function **sort-t** (Sec. 4.2) onto typed data:

$$\text{sort-td}(\text{dat}, \text{typ}) = \text{sort-t.typ}$$

Note in this place that although data were split onto several domain, we “glue” typed data into one domain. We can do so without losing a typing discipline, since data are coupled with types, and therefore the constructors of typed data may signalize errors whenever they receiving arguments of inappropriate types. As we will see in Sec. 6.4.1, this solution also leads to one carrier of value expressions denotations instead of many carriers such as, e.g., boolean expression denotations, integer expression denotations, etc. This decision simplifies our model.

Now, we proceed to the constructors of typed data. For each data constructor **da-co** we define a typed data constructor **td-co**, that “calls” (with one exception, for empty lists) the corresponding **da-co** and **ty-co**. Note that all our constructors are total functions which is possible due to the fact that **TypDatE** includes abstract errors.

#### Comparison constructors

|                    |                                   |                  |
|--------------------|-----------------------------------|------------------|
| <b>td-equal-in</b> | : <b>TypDatE</b> x <b>TypDatE</b> | ↦ <b>TypDatE</b> |
| <b>td-less-in</b>  | : <b>TypDatE</b> x <b>TypDatE</b> | ↦ <b>TypDatE</b> |
| <b>td-equal-re</b> | : <b>TypDatE</b> x <b>TypDatE</b> | ↦ <b>TypDatE</b> |
| <b>td-less-re</b>  | : <b>TypDatE</b> x <b>TypDatE</b> | ↦ <b>TypDatE</b> |
| <b>td-equal-te</b> | : <b>TypDatE</b> x <b>TypDatE</b> | ↦ <b>TypDatE</b> |

#### Arithmetic constructors for integers

|                  |                                   |                  |
|------------------|-----------------------------------|------------------|
| <b>td-add-in</b> | : <b>TypDatE</b> x <b>TypDatE</b> | ↦ <b>TypDatE</b> |
|------------------|-----------------------------------|------------------|

|                |                     |                   |
|----------------|---------------------|-------------------|
| td-subtract-in | : TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-multiply-in | : TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-divide-in   | : TypDatE x TypDatE | $\mapsto$ TypDatE |

#### Arithmetic constructors for reals

|                |                     |                   |
|----------------|---------------------|-------------------|
| td-add-re      | : TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-subtract-re | : TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-multiply-re | : TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-divide-re   | : TypDatE x TypDatE | $\mapsto$ TypDatE |

#### Text constructors

|            |                     |                   |
|------------|---------------------|-------------------|
| td-glue-tx | : TypDatE x TypDatE | $\mapsto$ TypDatE |
|------------|---------------------|-------------------|

#### List constructors

|              |                     |                   |
|--------------|---------------------|-------------------|
| td-empty-li  | : DatTypE           | $\mapsto$ TypDatE |
| td-put-to-li | : TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-head-li   | : TypDatE           | $\mapsto$ TypDatE |
| td-tail-li   | : TypDatE           | $\mapsto$ TypDatE |

#### Array constructors

|                 |                               |                   |
|-----------------|-------------------------------|-------------------|
| td-empty-ar     | : DatTypE x TypDatE           | $\mapsto$ TypDatE |
| td-change-in-ar | : TypDatE x TypDatE x TypDatE | $\mapsto$ TypDatE |
| td-get-from-ar  | : TypDatE x TypDatE           | $\mapsto$ TypDatE |

#### Record constructors

|                 |                                  |                   |
|-----------------|----------------------------------|-------------------|
| td-create-rc    | : Identifier x TypDatE           | $\mapsto$ TypDatE |
| td-put-to-rc    | : TypDatE x TypDatE x Identifier | $\mapsto$ TypDatE |
| td-change-in-rc | : TypDatE x Identifier x TypDatE | $\mapsto$ TypDatE |
| td-get-from-rc  | : TypDatE x Identifier           | $\mapsto$ TypDatE |

Let us show a few examples of the definitions of these constructors. All of them are transparent for errors (Sec. 2.9). In all cases constructors are defined according to a common scheme:

1. check if the argument typed data are not errors, and if they aren't then,
2. compute the resulting type by a type constructor, and if no error is signaled then,
3. compute the resulting data by a primary data constructor, and if no error is signaled then,
4. combine the computed type and data into a typed data.

If in 1, 2 or 3 an error is signaled, then this error becomes the final result. Let us illustrate this scheme by an example of the division of integers:

td-divide-in : TypDatE x TypDatE  $\mapsto$  TypDatE

```

td-divide-in.(tyd-1, tyd-2) =
  tyd-i : Error   $\rightarrow$  tyd-i           for i = 1, 2
  let
    (dat-i, typ-i) = tyd-i           for i = 1, 2
    typ             = ty-divide-in.(typ-1, typ-2)
  typ : Error   $\rightarrow$  typ
  let
    dat = da-divide-in.(dat-1, dat-2)
  dat : Error   $\rightarrow$  dat
  true       $\rightarrow$  (dat, typ)

```

In this definition, we refer to (call) two previously introduced constructors — a type constructors **ty-divide-in**, and a data constructor **da-divide-in**. First of them checks, if the arguments of **td-divide-in** are integers, and the other is responsible for all remaining checks.

A few other examples of the constructors of typed data are shown below. Note that two of them, like the coming one, get types as arguments.

```

td-empty-li : DatTypeE  $\mapsto$  TypDatE
td-empty-li.typ =
  let
    typ-l = ty-empty-li.typ
    typ-l : Error  $\rightarrow$  typ-l
    true  $\rightarrow$  (), typ-l

```

We recall that () denotes an empty tuple, and  $\text{ty-empty-li.typ} = ('L', \text{typ})$ . As we see, the empty list includes no data, but has a type. When we add a new typed data to such a list, its type must be  $\text{typ}$ . This rule is expressed in the following definition:

```

td-put-to-li : TypDatE x TypDatE  $\mapsto$  TypDatE
td-put-to-li.(tyd-e, tyd-l) =                                     -e – “element”, -l – “list”
  tyd-i : Error  $\rightarrow$  tyd-i          for i = e, l
  let
    (dat-i, typ-i) = tyd-i          for i = e, l
    new-typ-l = ty-put-to-li.(typ-e, typ-l)
    new-typ-l : Error  $\rightarrow$  new-typ-l
  let
    new-lis = da-put-to-li.(dat-e, dat-l)
  true  $\rightarrow$  (new-lis, typ-l)

```

When this operation is given a list that is homogeneous (all its elements are of the same type) and no error is encountered, then it builds a list which is homogeneous as well. Since we ensure this property for all the constructors of list-typed data, it follows that all reachable list-typed data are homogeneous.

Note that if  $\text{new-typ-l} \neq \text{typ-l}$ , then  $\text{new-typ-l} = \text{typ-l}$ , and therefore the type of the new list is  $\text{typ-l}$ . This definition in an unfolded (direct) form looks as follows:

```

td-put-to-li : TypDatE x TypDatE  $\mapsto$  TypDatE
td-put-to-li.(tyd-e, tyd-l) =                                     -e – “element”, -l – “list”
  tyd-i : Error  $\rightarrow$  tyd-i          for i = e, l
  let
    (typ-e, dat-e) = tyd-e
    (typ-l, dat-l) = tyd-l
    sort.typ-l  $\neq$  'L'  $\rightarrow$  'list-type expected'
  let
    ('L' typ-le) = typ-l                                             -le – „list element”
    typ-le  $\neq$  typ-e  $\rightarrow$  'types incompatible'
  true  $\rightarrow$  ((dat-e)@dat-l, typ-l)

```

The constructor that creates a “fresh” empty array is defined as follows

```

td-create-ar : DatTypeE x TypDatE  $\mapsto$  TypDatE
td-create-ar.(typ, tyd-i) =
  tyd-i : Error  $\rightarrow$  tyd-i
  let
    (dat, typ-i) = tyd-i
    typ-i  $\neq$  'integer'  $\rightarrow$  'array dimension must be an integer'
  let
    typ-a = ty-create-ar.(typ-e, dat)                                this constructor checks if dat > 0
    typ-a : Error  $\rightarrow$  typ-a
  true  $\rightarrow$  ([1/ $\Omega$ , ..., dat/ $\Omega$ ], typ-a)

```

The operation of getting an element from an array:

```

td-get-from-ar : TypDatE x TypDatE  $\mapsto$  TypDatE
td-get-from-ar.(tyd-ar, dat-in) =
  tyd-i : Error  $\rightarrow$  tyd-i          get an element of index dat-in from array tyd-ar
                                   for i = ar, in
  let
    (dat-i, typ-i) = tyd-i          for i = ar, in
    sort.typ-ar  $\neq$  'A'  $\rightarrow$  'array expected'
    sort.typ-in  $\neq$  'integer'  $\rightarrow$  'integer expected'
  let
    ('A', typ-el, int) = typ-ar
    (dat-ar, typ-ar) = tyd-ar
    (dat-in, 'integer') = dat-in
    dat-in  $\notin$  {1,...,int}  $\rightarrow$  'index beyond range'
  true  $\rightarrow$  (dat-ar.dat-in, typ-el)

```

At the end one more definition which “inherits” a decision from the level of types:

```

td-change-in-rc : TypDatE x Identifier x TypDatE  $\mapsto$  TypDatE
td-change-in-rc.(tyd-r, ide, tyd-e) =
  tyd-i : Error  $\rightarrow$  tyd-i          change in record tyd-r at attribute ide for tyd-e
                                   for i = r, e
  let
    (dat-i, typ-i) = tyd-i          for i = r, e
    typ = ty-change-in-re.(typ-r, ide, typ-e)
  typ : Error  $\rightarrow$  typ
  let
    new-rec = da-change-in-re.(dat-r, ide, dat-e)
  true  $\rightarrow$  (new-rec, typ-r)

```

Here we assume that the corresponding data-constructor is the following:

$da\text{-}change\text{-}in\text{-}rc.(dat\text{-}r, ide, dat\text{-}e) = dat\text{-}r[ide/dat\text{-}e]$

The inherited decision is implicit in `ty-change-in-re` and concerns the fact that if we assign new data to an attribute of a record, then the new type must be identical with the previous one. Consequently the type of the record does not change.

In this place one methodological remark may be in order. In building our constructors of typed data we first built corresponding constructors of data, then of types and finally of typed data. Technically it might be simpler and shorter to define typed data constructors in one step. However, we decided to do it in a stepwise way, since it is part of the following “technological line”:

1. data,
2. types,
3. typed data
4. values,
5. value expression denotations,
6. value expression syntax.

In each of these steps we concentrate on a different stage of the design of our language, i.e., on a different aspect of this language.

## 4.4 Values, references, objects, deposits and types

Two major concepts that we discuss in this section are *values* and *references*. As already announced in Sec. 4.3, there are two categories of values: typed data and *objects*. Values may be:

- returned by value expressions (Sec. 6.4.1),
- assigned to references in deposits (in states, cf. Sec. 5.3),

- passed to procedures as the values of actual value-parameters (Sec. 6.6.3.4).

References are pairs consisting of a *token*, representing some memory location, and a *profile*. The profile describes the *usability* and the *visibility* of the reference (Sec. 5.4). The former determines properties of values which can be stored under this reference, the latter — the rules of accessing them.

The domains of values and references, as well as their related domains, are defined by the following equations:

|     |             |                                      |              |
|-----|-------------|--------------------------------------|--------------|
| val | : Value     | = TypDat   Object                    | values       |
| obj | : Object    | = Objecton x ObjTyp                  | objects      |
| obn | : Objecton  | = Identifier $\Rightarrow$ Reference | objectons    |
| typ | : ObjTyp    | = Identifier                         | object types |
| ref | : Reference | = Token x Profile                    | references   |
| tok | : Token     | = ... (e.g. memory locations)        | tokens       |
| prf | : Profile   | = Type x OriTag                      | profiles     |
| typ | : Type      | = DatTyp   ObjTyp                    | types        |
| ota | : OriTag    | = Identifier   {\$}                  | origin tags  |
| dep | : Deposit   | = Reference $\Rightarrow$ Value      | deposits     |

An *object* is a pair (obn, typ) that consists of an *objecton* and an *object type*. The latter is an identifier which is supposed to be a name of a class (Sec. 5.2). Objects may be said, therefore, to be typed objectons.

An objecton may be regarded as a memory structure whose fields, i.e., references, are bound to identifiers that we shall call *attributes*.

A *reference* ref = (tok, prf) is a pair consisting of:

- a *token* tok, that represents a memory location,
- a *profile* prf = (typ, ota), that determines the way in which ref may be used:
  - type typ determines the *usability* of ref by indicating the type of values that may be assigned to this reference in deposits,
  - *origin tag* ota determines the *visibility status* of ref — if it is \$ then the reference is public and otherwise it is private (details in Sec. 5.4.3); we assume that \$ does not belong to Identifier, and we call it a *public-visibility tag*.

*Deposits* describe memory contents, since they assign values to references. We will make sure that references in the domain of each deposit that can be built carry distinct tokens.

In the sequel (Sec. 5.3) each memory state will carry an objecton and a deposit. If a reference assigned to an identifier in an objecton does not belong to the domain of (the current) deposit, then the identifier is said to be *declared* but *not initialized*, and its reference is said to be a *dangling reference*.

We introduce a special notation and terminology to be used in talking about objects. Consider an objecton obn, a deposit dep, and an identifier ide. We write then:

ide  $\rightarrow$  ref and we say that *ide points to ref*, if obn.ide = ref,  
 ref  $\rightarrow$  val and we say that *ref points to val*, if dep.ref = val,

Note that we are not talking here about a reference pointing to a location (which is a typical use of references or pointers in programming) but about an identifier pointing to a reference which in turn denotes the memory location at which the identifier's value is being stored. Given an identifier ide in a context of an objecton obn and a store sto (i.e., in the context of a memory state; see Sec.5.3), three situations are possible:

|   |   |
|---|---|
| ide $\rightarrow$ ref $\rightarrow$ val | a standard situation where ide has been declared and initialized; in this case we say that val is the (current) <i>value</i> of ide,                                    |
| ide $\rightarrow$ ref                   | ide has been declared but not initialized; i.e., ref is a <i>dangling reference</i> ,   |
| ref $\rightarrow$ val                   | no identifier points to ref in obn; in this case we say that ref is an <i>orphan reference</i> in obn; objectons with orphan references may appear for instance when we |

create a local initial store of a procedure call (Sec. 6.6.3.4), and when we return from a local terminal store of a procedure call to a global terminal store (Sec. 6.6.3.5).

Note, however, that a reference **reg** which is orphan in **obn** may be pointed in an objecton of an object which is pointed (directly or indirectly) by **ref**. To be an orphan is, therefore, a property of a reference in a context of a concrete objecton.

References that belong to the range of an objecton, are said to be *carried* by this objecton, and by objects that include this objecton.

The profile, the type, and the origin tag of a reference are also said to be, respectively, the profile, the type, and the origin tag of an attribute that points to this reference.

We extend to values the function that returns the sort of a typed data:

```
sort-val.val =
  val : TypDat → sort-td.val
  val : Object → 'object'
```

In a certain sense objectons may be seen as “multireferences” because each of their attributes points to a reference. Note also that these references may point to other objects that carry further references, etc. Consequently, objects may represent nested structures. We see such a situation in Fig. 4.4-1, where **no1**, **ob1**,... are attributes, **nr1**, **or1**,... are references, **A** is a metaname of an objecton, **B** and **C** are metanames of objects, **ClassB** and **ClassC** are names of classes (identifiers), i.e., are the types of corresponding objects.

If we fix a deposit **dep**, then two objectons **obn-1** and **obn-2** are said to be *sibling objectons* if:

- they have a common set of attributes,  $\text{dom.obn-1} = \text{dom.obn-2}$ ,
- their sets of tokens are disjoint, i.e.,  
 $\{\text{tok} \mid (\exists \text{ide}) (\exists \text{prf}) (\text{tok}, \text{prf}) = \text{obn-1.ide}\} \cap \{\text{tok} \mid (\exists \text{ide}) (\exists \text{prf}) (\text{tok}, \text{prf}) = \text{obn-2.ide}\} = \{\}$
- corresponding attributes have equal profiles, i.e.,  
 if  $\text{obn-i.ide} = (\text{tok-i}, \text{prf-i})$  for  $i = 1;2$ , then  $\text{prf-1} = \text{prf-2}$ ,
- the objectons of the corresponding inner objects are sibling objectons, i.e.,  
 if  $\text{dep}(\text{obn-i.ide})$  for  $i = 1;2$  are both objects, then their objectons are siblings.

If we imagine objectons as trees, then sibling objectons have the same structures, and the same labels of nodes (attributes), but differ tokens and possibly different non-object values at their leaves. Two objects are said to be *sibling objects*, if they are of the same type and their objectons are siblings.

For a fixed deposit **dep** two objectons **obn-1** and **obn-2** are said to be *twin objectons* if:

- they are sibling objectons,
- if  $\text{dep}(\text{obn-i.ide})$  for  $i = 1;2$  are not objects, then they are equal,
- if  $\text{dep}(\text{obn-i.ide})$  for  $i = 1;2$  are both objects, then their objectons are twin objectons.

In other words, twin objectons are siblings with the same values at their leaves. Two references  $(\text{tok-i}, \text{prf-i})$  for  $i = 1,2$  are *twin references*, if  $\text{tok-1} \neq \text{tok-2}$  and  $\text{prf-1} = \text{prf-2}$ .

In this section we shall not define any constructors of objects since objects will be built exclusively by functional procedures called *object constructors* (Sec. 6.6.5). To the category of types we add a constructor of object types, which from a set-theoretical perspective is an identity function, but from an algebraic perspective it is not, because it “makes identifiers to be object types”.

```
ty-create-ot : Identifier → ObjTyp
ty-create-ot.ide = ide
```

Such an algebraic constructor is called an *insertion*.



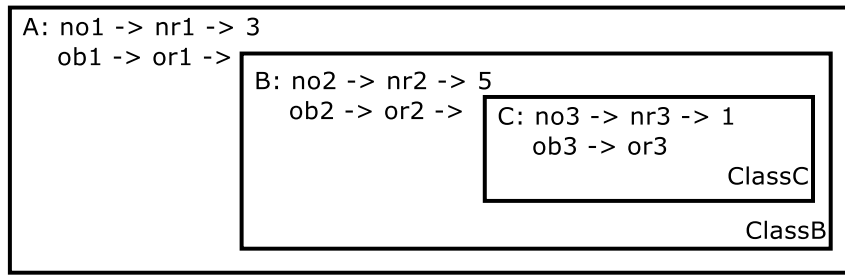


Fig. 4.4-1 A structure view of an objecton

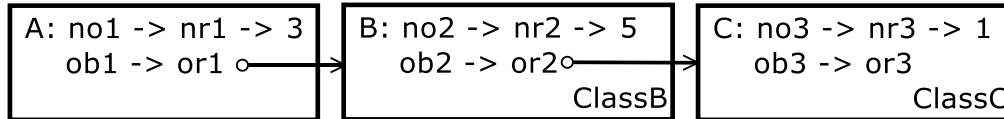


Fig. 4.4-2 A graph view of an objecton

It is to be noted that structured types, i.e. array-, list- and record types always belong to the category of datatypes. We do not introduce constructors to build structured types involving object types, such as, e.g., types of lists or arrays of objects, which is a consequence of the decision not to deal with structured values such as, e.g., lists or arrays of objects. We refrain from discussing these options just for the sake of simplicity and brevity. As we are going to see in Sec. 6.6.5.3, new objects will be built exclusively by *object constructors*, and by instructions that modify earlier constructed object.

Nevertheless, we can build objects that may be colloquially called “lists of objects”, but their “list nature” is just a way of seeing them.

Consider an objecton in Fig. 4.4-1. Its visualization will be referred to as *structure view* of this objecton. An alternative to it is a *graph view* — in this example a *list view* — shown in Fig. 4.4-2., but it is not a list in the sense of Sec.4.1. In our model we only have values that are lists, but we do not have lists of values.

If an object **B** is assigned to an attribute of an object **A**, as in our example, then we say that **B** is an *inner object of depth 1* of **A**. The inner objects of **B** will be also regarded as inner objects of **A**, but of a higher depths and so on. The attributes of **A** will be called *surface attributes* of **A**, whereas all attributes of **B** and **C** will be called *deep attributes* of **A**.

In the sequel we shall carefully distinguish between:

- *object attribute*, that is analogous to *integer attribute* and that is an attribute whose value is an object,
- and an *attribute of an object*, or *object's attribute*, that is an attribute in the objecton of an object.

As we are going to see, it may be convenient to regard a value as a pair consisting of an element that we shall call *the core of the value*, and a type:

val : Value = Core x Type  
cor : Core = Data | Objecton



## 5 CLASSES AND STATES

### 5.1 Classes intuitively

Classes may be regarded as collections of tools used to create and modify objects. They carry three categories of tools:

1. types (maybe none),
2. methods (maybe none), which are either signatures of procedures, or so called pre-procedures, and which include two subcategories:
  - a. imperative pre-procedures, functional pre-procedures and their corresponding signatures,
  - b. object pre-constructors and their signatures,
3. one objecton (maybe empty) used as a pattern for all objects generated from this class.

In this place we have to explain that pre-procedures and pre-constructors constitute sort of “raw materials” of future procedures and object constructors that will be created in a “single step” once all classes have been declared (Sec. 6.7.6). This construction is necessary to describe the mechanism of procedure creation from mutually recursive pre-procedures declared in different classes.

Let’s forget for a moment about types and methods, and concentrate on objectons carried by classes. Consider the following class declaration written in an anticipated syntax<sup>30</sup> of **Lingua**.

```
class CartesianPoint
  let abscissa = 2.15 be real and public tel;           a declaration of an attribute
  let ordinate be real and private tel                 a declaration of an attribute
ssalc
```

The fact that **abscissa** is initialized to 2.15 only means that if we generate an object directly from the class, then **abscissa** will be initialized to 2.15 but later we can change its value. In turn the attribute **ordinate** may be (but do not need to) left not initialized. Consequently an object of type **CartesianPoint** may be of the form (in the context of a deposit):

```
'abscissa' → (ab-tok, ('real', TT), '$') → (2.15, ('real', TT))
'ordinate' → (or-tok, ('real', TT), 'CartesianPoint')
                                     'CartesianPoint'
```

or of the form

```
'abscissa' → (ab-tok, ('real', TT), '$') → (3.16, ('real', TT))
'ordinate' → (or-tok, ('real', TT), 'CartesianPoint') → (4.75, ('real', TT))
                                     'CartesianPoint'
```

In the first case the reference of ‘ordinate’ is dangling which expresses the fact that this attribute has been declared but not initialized. In the second case it has been initialized to a real value. Since this attribute has been declared as private its origin tag is ‘CartesianPoint’.

In our example both objects of class **CartesianPoint** are of the same shape. The situation complicates, when we introduce recursion to the definitions of a class. Let’s consider the following class declaration written again in an anticipated syntax.

```
class ListNode
```

<sup>30</sup> In a general case declarations of variables indicate types and privacy status of these variables.

```

let no = 23 be integer and public tel;
let next be ListNode and public tel

cons ConstructObject(val number as integer, node as ListNode return ListNode)
    no    := number + 1;
    next  := node
snoc

proc BuildCircularList()
    let i be integer tel;
    let node be ListNode tel;
    i := 1;
    while i <= 3
        do
            node := ListNode.ConstructObject(i, node);
            i    := i+1
        od;
    node.next.no    := 11
    node.next.next  := node;
corp

ssalc;

```

the end of class declaration

ListNode.BuildCircularList()

In this case our class declaration includes three declarations:

1. of an objecton's pattern,
2. of a special-purpose functional procedure called an *object constructor*,
3. of an imperative procedure which calls the object constructor and modifies global memory states.

Below this declaration we see a call an object constructor `BuildCircularList()` declared in class `ListNode`.

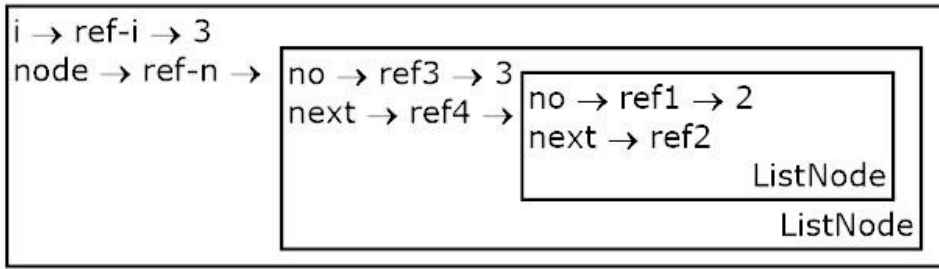
In this example the attribute `next` is of the type of the class which is just being declared. The execution of `BuildCircularList()` generates a sequence of objectons shown below, where the first objecton results from the execution of the declaration of local attributes of this procedure, and belongs to an initial local state of the call rather than to the created object. In the graphical representations of our objectons we show not only their references but also the values assigned to these references in the corresponding deposits.

|  |
|--|
| $i \rightarrow \text{ref-i} \rightarrow 1$<br>$\text{node} \rightarrow \text{ref-n}$ |
|--|

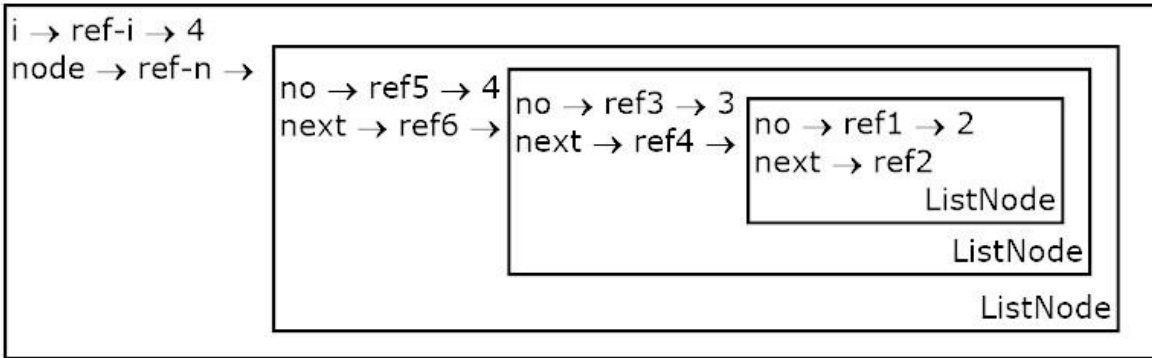
In the next step our procedure enters the `while` loop, and there calls the object constructor `ConstructObject` and passes to it two actual value parameters: `i` of value 1, and `node` with a dangling reference. The object constructor builds an object by copying (with new references) and modifying the objecton of the class. Then the new object is assigned to `node` and the value of `i` is augmented by one.

|  |  |          |
|--|--|----------|
| $i \rightarrow \text{ref-i} \rightarrow 2$<br>$\text{node} \rightarrow \text{ref-n} \rightarrow$ <table border="1"> <tr> <td> <math>\text{no} \rightarrow \text{ref1} \rightarrow 2</math><br/> <math>\text{next} \rightarrow \text{ref2}</math> </td> </tr> <tr> <td>ListNode</td> </tr> </table> | $\text{no} \rightarrow \text{ref1} \rightarrow 2$<br>$\text{next} \rightarrow \text{ref2}$ | ListNode |
| $\text{no} \rightarrow \text{ref1} \rightarrow 2$<br>$\text{next} \rightarrow \text{ref2}$   |  |          |
| ListNode   |  |          |

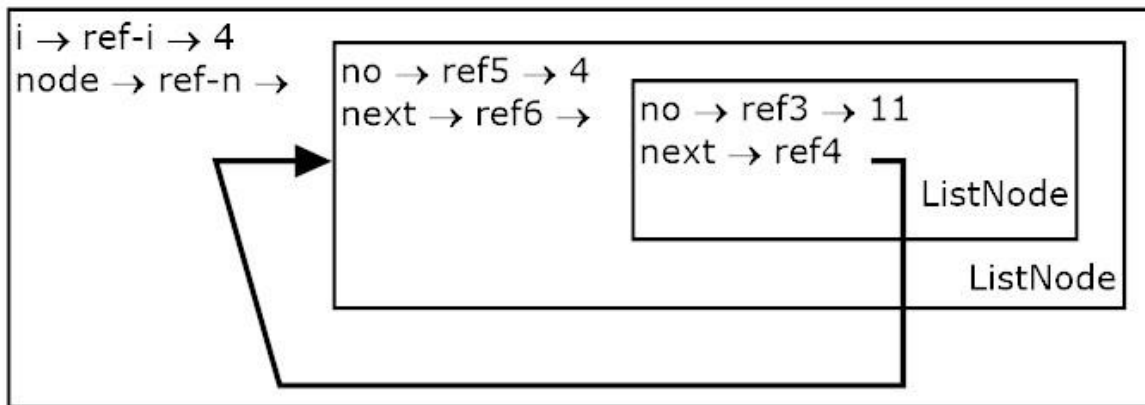
In the following step we assign to `node` a modified class objecton where the formerly created object is assigned to `next`:



This action is repeated thus producing the following objecton. Our program exits the loop.



In the last step our program performs two last assignments which in the deepest object modifies the value of **no**, and redirects the reference of the deepest **next** to the surface objecton. In this way we have constructed a circular object. The rules of building objects from classes are formalized in Sec.6.6.5.



At the end of this section let's list assumptions about classes and objects in our model that we shall adopt to make it possibly free from technical complications:

1. We do not introduce neither packages nor compilation units.
2. Classes do not include inner classes.
3. As a consequence of 1. and 2. all classes are public.
4. Classes do not contain not-replicable attributes.
5. Types and methods are declared exclusively in classes and are public.
6. A class once declared is never changed, but can be copied and then modified to build a new class (heritage).
7. Objects are created exclusively by object constructors that replicate class objectons.
8. Some attributes of object may be private; for such attributes, if we wish to provide an external access to them we have to declare appropriate **getters** and/or **setters** in the corresponding class.

## 5.2 Classes formally

By a *class* we shall mean a tuple consisting of four elements: an identifier, two mappings (possibly empty) — a *type environment*, and a *method environment* — and one objecton (possibly empty):

|                               |  |                     |
|-------------------------------|--|---------------------|
| $\text{cla} : \text{Class}$   | $= \text{Identifier} \times \text{TypEnv} \times \text{MetEnv} \times \text{Objecton}$ | classes             |
| $\text{tye} : \text{TypEnv}$  | $= \text{Identifier} \Rightarrow \text{Type} \mid \{\emptyset\}$                       | type environments   |
| $\text{mee} : \text{MetEnv}$  | $= \text{Identifier} \Rightarrow \text{Method}$  | method environments |
| $\text{met} : \text{Methods}$ | $= \text{ProSig} \mid \text{PrePro}$   | methods             |

where  $\emptyset$  is a special element called a *pseudotype*. The domains **ProSig** of *procedure signatures*, and **PrePro** of *pre-procedures*, will be defined in Sec. 6.6. Each class is, therefore, a tuple of four elements:

$(\text{ide}, \text{tye}, \text{mee}, \text{obn})$ .

By an *empty class* we mean a class where all three mappings are empty:

$(\text{ide}, [], [], [])$ .

Identifiers bound to types will be called *constants*, since their values, once assigned to them, are never changed. The first element of a class, the identifier, is called an *internal name of a class*. We will see why we need these internal names in Sec. 6.7.4.2, where we describe an action of adding a new attribute to (the objecton of) a class.

### 5.3 Stores and states

The domain of states is defined as follows:

|                                 |   |                                  |
|---------------------------------|---|----------------------------------|
| $\text{sta} : \text{State}$     | $= \text{Env} \times \text{Store}$  | states                           |
| $\text{env} : \text{Env}$       | $= \text{ClaEnv} \times \text{ProEnv} \times \text{CovRel}$   | environments                     |
| $\text{cle} : \text{ClaEnv}$    | $= \text{Identifier} \Rightarrow \text{Class}$  | class environments               |
| $\text{pre} : \text{ProEnv}$    | $= \text{ProInd} \Rightarrow \text{Procedure}$  | procedure environments           |
| $\text{pri} : \text{ProInd}$    | $= \text{Identifier} \times \text{Identifier}$  | procedure indicators             |
| $\text{sto} : \text{Store}$     | $= \text{Objecton} \times \text{Deposit} \times \text{OriTag} \times \text{SetFreTok} \times (\text{Error} \mid \{\text{'OK'}\})$ | stores                           |
| $\text{cov} : \text{CovRel}$    | $= \text{Sub}((\text{DatTyp} \times \text{DatTyp}) \mid (\text{ObjTyp} \times \text{ObjTyp}))$                                    | covering relations               |
| $\text{sft} : \text{SetFreTok}$ | $= \dots$   | an infinite set of (free) tokens |

The environment of a state carries classes, procedures<sup>31</sup> (Sec. 6.7.6), and covering relations<sup>32</sup> (Sec. 5.4.2), and the store carries the rest. Classes and types declared in them are going to be public, whereas their methods and attributes are going to be private<sup>33</sup>. Also the attributes carried by objectons of stores will be private. The meanings of “public” and “private” are explained in Sec. 5.4.3.

Identifiers bound to values (though references) in state objectons, but not in class objectons, will be called *variables*, since their values may be modified.

If a class is assigned to an identifier in a class environment, then we say that this identifier *points to* this class. In an analogous way we talk about identifiers pointing to types in the type environments of classes, and to pre-procedures in procedure environments.

Identifiers that point to classes in states are called the *external names* of these classes, and the corresponding classes are said to be *declared* in **sta**. A little later we shall assume that in well-formed states external names of classes must coincide with their internal names (Sec. 5.2).

Attributes that appear in objectons of classes are called *class's attributes*. A surface attribute of the objecton of a store is traditionally called a *variable* in this store and state respectively.

An object of the form  $(\text{obn}, \text{MyClass})$ , where **obn** has been created by an object constructor (see Sec. 6.6.5.2) from the objecton of **cle.MyClass**, is said to be an *object of class MyClass*.

<sup>31</sup> We recall that classes carry pre-procedures rather than procedures. The difference between these two concepts is explained in Sec. 6.6.

<sup>32</sup> The decision of putting covering relations in environments is technically not especially relevant. We just decided to “keep them in the same place”, where we keep classes and their types.

<sup>33</sup> This decision has an editorial character and serves the technical simplification of our model.

The covering relations  $\text{COV}$  between types will be used to describe a *usability regime* defined in Sec. 5.4.2.

The origin tag that appears in a store is called the *origin tag of the store*, and of the hosting state as well. Its role will be explained in Sec. 5.4.3, where we shall talk about a *visibility regime*.

The sets of free tokens will be used to provide "fresh" (not used) tokens, for the declarations of value variables and the constructors of new objects. For that sake we assume the existence in our model of a function:

$\text{get-tok} : \text{SetFreTok} \mapsto \text{Token} \times \text{SetFreTok}$   
 $\text{get-tok.sft} = (\text{tok}, \text{sft} - \{\text{tok}\})$  such that  $\text{tok} : \text{sft}$

An objecton  $\text{my-obn}$  is said to be *well-formed* in a state  $\text{sta} = ((\text{cle}, \text{pre}, \text{cov}), (\text{obn}, \text{dep}, \text{ota}, \text{sft}, \text{err}))$ , if:

1. for any attribute  $\text{ide}$ , if  $\text{obn.ide} = !$ , and  $\text{dep}(\text{obn.ide}) = !$ , then:  
 $\text{obn.ide} \mathbf{VRA.cov} \text{dep}(\text{obn.ide})$  — value by reference acceptability (see Sec. 5.4.2),
2. all inner objectons of  $\text{obn}$  are well-formed in  $\text{sta}$ .

A class  $(\text{ide}, \text{tye}, \text{mee}, \text{obn})$  is said to be *well-formed* in a state, if

1.  $\text{obn}$  is well-formed in this state,
2. for every reference  $(\text{tok}, (\text{typ}, \text{ota}))$  in  $\text{obn}$ , its origin tag  $\text{ota}$  is either  $\$$  or  $\text{ide}$ <sup>34</sup>.

A state  $\text{sta} = ((\text{cle}, \text{pre}, \text{cov}), (\text{obn}, \text{dep}, \text{ota}, \text{sft}, \text{err}))$  said to be *well-formed*, if:

1.  $\text{obn}$  is well formed in  $\text{sta}$ ,
2. external names of all classes declared in  $\text{cle}$  coincide with their internal names,
3. all surface and inner objects in  $\text{obn}$  are of types that are the names of classes declared in  $\text{cle}$ ,
4. all classes declared in  $\text{cle}$  are well-formed,
5.  $\text{sft}$  includes only such tokens that do not appear in references bound in  $\text{dep}$ ,
6. every identifier appearing in a state, appears in it only once; e.g., if an identifier is a variable, it can't be at the same time a type constant or a class name<sup>35</sup>.

As we see, the well-formedness of states is mainly about typing. In the sequel, we shall ensure that the states appearing in the executions of our programs are well-formed. By:

### WfState

we denote the sets of all well-formed states. For technical convenience we define the following auxiliary functions:

$\text{error} : \text{Store} \mapsto \text{Error} \mid \{\text{'OK'}\}$        $\text{error} : \text{State} \mapsto \text{Error} \mid \{\text{'OK'}\}$   
 $\text{error}(\text{obn}, \text{dep}, \text{ota}, \text{sft}, \text{err}) = \text{err}$        $\text{error}(\text{env}, \text{sto}) = \text{error.sto}$

Formally we may assume that the function  $\text{error}$  is defined on the union  $\text{Store} \mid \text{State}$ . In the same spirit we define next two functions:

$\text{is-error} : \text{Store} \mapsto \text{Boolean}$        $\text{is-error} : \text{State} \mapsto \text{Boolean}$   
 $\text{is-error.sto} =$        $\text{is-error}(\text{env}, \text{sto}) = \text{is-error.sto}$   
 $\text{error.sto} \neq \text{'OK'} \rightarrow \text{tt}$   
 $\text{true} \rightarrow \text{ff}$

Again, not quite formally, we define a function on the domain  $(\text{State} \times \text{SetFreTok}) \mid (\text{State} \times \text{Error})$ :

$\blacktriangleleft : \text{State} \times \text{Error} \mapsto \text{State}$   
 $(\text{env}, (\text{obn}, \text{dep}, \text{ota}, \text{sft}, \text{err})) \blacktriangleleft \text{new-err} = (\text{env}, (\text{obn}, \text{dep}, \text{sft}, \text{ota}, \text{new-err}))$   
 $\blacktriangleleft : \text{State} \times \text{SetFreTok} \mapsto \text{State}$

<sup>34</sup> This definition is the main cause why we have introduced internal names of classes as the elements of classes (see also Sec. 6.7.4.2). An alternative solution might be to talk about the well-formedness of classes only in the context of states, that, in our opinion, would be less elegant.

<sup>35</sup> Mathematically this assumption is not necessary, but we have taken this engineering decision for the sake of a better readability and clarity of programs.

$(env, (obn, dep, ota, sft, err)) \blacktriangleleft new-sft = (env, (obn, dep, new-sft, ota, err))$

We also assume that this function will be applicable to stores in an obvious way. We shall use a function:

$declared : Identifier \times State \mapsto \{tt, ff\}$

to protect us against double declaration of identifiers. We assume that **declared** “detects” the appearance of an identifier in any role — i.e., as a variable, an attribute, a type, a method or a class — and at any level, i.e., in a state object or in an object of any object at any depth.

By an *empty state* we shall mean every state of the form  $(([ ], [ ], Ld-cov), ([ ], [ ], '\$', \{ \}, 'OK'))$  where *Ld-cov* is a covering relation defined by language designer (see Sec. 5.4.2). As is easy to check, empty state is well formed.

## 5.4 Two regimes of handling items

### 5.4.1 An overview

By an *item*, we shall mean a value (both — typed data and objects), a reference, a type, a method (Sec. 6.6), or a class. All items are storable, and we access them through *indicators* that are tuples of identifiers. To indicate a class, we need one identifier, to indicate a type or a procedure, we need two identifiers — one for a class plus one for the type/procedure itself — to indicate a value or a reference of a variable, we need one identifier, but in the case of object attributes, we may need more identifiers, if such an attribute is located at a deep level of an object.

The principles of accessing and using items will be described by two systems of rules that we shall call *handling regimes*:

- A *usability regime* is described using the types of values and the profiles of references. It serves the purpose of deciding which values can be “sent” to an operator as its arguments or can be assigned to a reference in a deposit. For instance, we can’t “send” real numbers to an integer. Technically, the usability regime is built into the denotations of expressions, assignment instructions, variable- and attribute declarations, procedure declarations, and procedure calls.
- A *visibility regime* is described using the origin tags of references and states and serves the purpose of deciding which item indicators are accessible at a given stage of program execution; e.g., we shall assume that private attributes of a class will be visible exclusively in the bodies of procedures declared in this class. Technically, a reference, to be visible in a state, must have the origin tag identical to the tag of the state (cf. assignment instructions in Sec. 6.4).

Note that pre-procedures are not regarded as items. They will not be accessible from syntactic level, and will constitute sort of “raw components” used in building procedures. This technique, which is explored in Sec. 6.7.6, has been adopted to describe the execution of mutually recursive procedures declared in different classes. Procedure declarations included in the declarations of classes will first assign pre-procedures to procedure names in method environments of classes, and later a special mechanism of *procedure opening* (Sec. 6.7.6) will assign procedures to their indicators in procedure environments of states. Although formally procedure declarations in classes build pre-procedures, we shall talk, for simplicity, about procedures declared in classes.

Note a significant differences between two described regimes — usability is a property of values, indicated by their types, whereas visibility is a property of the indicators of items, and when these items are values, the visibility of their indicators is encoded in the profiles of references that they point to. We may also colloquially say that a locally declared procedure is visible only in a local state, but precisely speaking, what may be seen or not is the indicator of a procedure rather than a procedure itself. Of course,

In two following sections we give a birds-eye view to the ideas of our handling regimes, to be later incorporated in our model.



## 5.4.2 Usability regime

Basic rules of usability regime are the following

1. Every value includes a type, and every reference includes a profile consisting of a type and origin tag.
2. If a value is going to be assigned to an attribute (via its reference), by a declaration (Sec. 6.7.2), or by an assignment instruction (Sec.6.4), or by a parameter passing mechanism of a procedure (Sec.6.6.3.4), then the type of the attribute must *accept* the type of the value. The concept of type acceptance is explained below.
3. If a function (operation) is applied to its arguments, then the types of arguments “expected” by the function must *accept* the types of the current arguments.

To formalize the concept of type acceptance we return to the notion of a *covering relations* with the following domain (Sec. 5.3):

$$\text{cov} : \text{CovRel} = \text{Sub}((\text{DatTyp} \times \text{DatTyp}) \mid (\text{ObjTyp} \times \text{ObjTyp}))$$

The fact that  $(\text{typ-1}, \text{typ-2}) : \text{cov}$  will be also written as **typ-1 cov typ-2**, and will be said that **typ-1 covers**, or *accepts* **typ-2**. As we see, a data type may cover only another data type, but not an object type, and vice versa. The typesetting of **cov** in bold is just a “meta-syntactic sugar” to make some meta-formulas easier to read.

We assume that each covering relation **cov** will be partly defined by a language designer, and partly by a programmer. Consequently it will be a union of two (disjoint) relations:

$$\text{cov} = \text{Ld-cov} \mid \text{Pr-cov}$$

where

1. **Ld-cov** is a component defined by a language designer, i.e. available in all programs of a given language,
2. **Pr-cov** is a component defined by a programmer, i.e. available exclusively in the program where it has been established.

For instance, a language designer may decide that a data type ‘integer’ covers data type (‘small-integer’), and a programmer — that an object type ‘employee-type’ covers object type ‘accountant’.

Now, for every covering relation **cov** we define two induced *acceptability relations*:

$$\mathbf{TTA.cov} \subseteq \text{Type} \times \text{Type} \quad \text{type-by-type acceptability relation}$$

$$\mathbf{VRA.cov} \subseteq \text{Reference} \times \text{Value} \quad \text{value-by-reference acceptability relation}$$

The first of them is a completion of **cov** to a reflexive and transitive relation, which means that **TTA.cov** is the least relation on types such that

- (1)  $\text{cov} \subseteq \mathbf{TTA.cov}$ ,
- (2)  $(\text{typ}, \text{typ}) \subseteq \mathbf{TTA.cov}$  for every  $\text{typ} : \text{Type}$ ,
- (3) if  $(\text{typ1}, \text{typ2}), (\text{typ2}, \text{typ3}) : \mathbf{TTA.cov}$  then  $(\text{typ1}, \text{typ3}) : \mathbf{TTA.cov}$ .

The second induced relation is derived from the former and has only technical character:

$$(\text{tok}, (\text{typ-r ota})) \mathbf{VRA.cov} (\text{cor}, \text{typ-v}) \quad \text{iff} \quad \text{typ-r } \mathbf{TTA.cov} \text{ typ-v}$$

In Sec. 5.3 we have assumed that the relationship  $\text{ref } \mathbf{VRA.cov} \text{ val}$  must be satisfied in all well-formed states.

In defining the denotational level of **Lingua** we shall give our programmers a tool for the creation of covering relations by enriching current relations by new pairs. For this sake we assume the existence in our model of a function of enrichment with the following signature:

$$\text{enrich-cov} : \text{CovRel} \times \text{Type} \times \text{Type} \mapsto \text{CovRel} \mid \text{Error}$$

and the following definitional scheme:

$$\begin{aligned} \text{enrich-cov}(\text{cov}, \text{typ-1}, \text{typ-2}) = \\ \text{typ-1} : \text{ObjTyp} \text{ and } \text{typ-2} \neq \text{ObjTyp} \quad \rightarrow \text{'typ-2 must be an object type'} \end{aligned}$$



|                |                     |  |
|----------------|---------------------|--|
| typ-2 : ObjTyp | and typ-1 /: ObjTyp | → 'typ-1 must be an object type'       |
| typ-1          | TTA.cov typ-2       | → 'redundant definition' <sup>36</sup> |
| other cases    |                     | → other error signals                  |
| true           |                     | → cov   {(typ-1, typ-2)}               |

We leave the “other cases” not specified to avoid going into too many technical details. An example of a pair of types that may be rejected by `enrich-cov` may be `(typ, ('A', typ))`.

Note that our constructor does not check if object types added to `COV` are carrying the names of declared classes. Such a check must refer to a state, and therefore will be introduced at the level of denotations in Sec. 6.7.5.

### 5.4.3 Visibility regimes

From a programmer’s perspective, visibility rules explain in which programming contexts a given item indicator may be used (is visible). E.g. we will set a rule that private attributes of a class may be referred to exclusively in the bodies of procedures declared in this class. At the same time, items locally declared in the body of a procedure will be visible exclusively in this body.

In our model of object-oriented languages we will have two orthogonal “dimensions” of *visibility statuses*:

- **procedure-dependent visibility**: all items locally declared in procedure bodies will be visible exclusively in these bodies,
- **class-dependent visibility**: selected items in classes may be declared as *private*.

Let’s start from the former, and let’s anticipate an assumption later formalized in Sec. 6.3, that every program in our model consists of a (possibly composed) declaration followed by a (possibly composed) instruction. This assumption does not limit the expressiveness of programs, but considerably simplifies our model. Under this assumption all global instructions of a program, i.e. all instructions except local instructions of procedure calls, operate on a common *global environment*, and a common *global-state objecton*. It is so, since instructions may only change values assigned to references in deposits.

Assume now that we call an imperative procedure in some current state which we shall call *initial global state* and which consists of a *global environment* and an *initial global store* which includes the common global-state objecton and some current deposit:

`ig-sta = (g-env, ig-sto),`

In our model the execution of a procedure call consists of three steps (to be described formally in Sec. 6.6.3):

- First, we create an *initial local state* `il-sta = (g-env, il-sto)`, that consists of a global environment, and an *initial local store*. The latter binds (in the objecton) only formal parameters. At the same time it inherits the whole deposit from the global store. Formal reference-parameters point directly to the references of actual-reference parameters, and the references of the remaining global variables become orphans (Fig. 6.6-2 in Sec. 6.6.3.4)
- Next, we execute the body of the procedure thus transforming the initial local state into a *terminal local state* `tl-sta = (tl-env, tl-sto)`. Since the bodies of procedures may be arbitrary programs, in the course of their executions local classes, procedures and variables may be declared.
- At the end of the call, we exit from the call, and create a *terminal global state* `tg-sta = (g-env, tg-sto)`, where we regain the global environment, and global deposit (Sec. 6.6.3.6). At this stage all locally declared classes, procedures, and variables cease to exist, and therefore are no more visible.

Note now, that in the above anticipation of the mechanism of imperative procedures we have made three important decisions concerning global visibility:

---

<sup>36</sup> Mathematically we could have assumed that in this case the enrichment operation returns an unchanged `COV` relation. However, if a programmer tries to add a pair of types that is already in `COV`, then they probably do it by mistake, and therefore such fact should be signalized by the system.

- all global classes are visible in all local states of procedure calls, since they are declared in the global environment,
- the references of all actual reference parameters are visible in local stores,
- all locally declared classes and variables cease to exist after the termination of the call.

All these decisions have an engineering character, since from a mathematical perspective, we could have decided differently, e.g. that locally declared items remain visible after exiting a procedure call, or that only some of the globally declared items are visible locally.

As we are going to see in Sec. 6.7, our mechanism of public visibility is even more complicated, since in procedure declarations we accept an *anticipated visibility* of procedures that haven't been declared yet. At the same time, however, in the case of types we require an *ex post visibility*, which means that a type must be declared to be visible.

Let us proceed now to visibility statuses. We assume — again for the simplicity of our model — that private may be only the attributes of classes and objects, if they are declared to be so, whereas all other items are always public.

*Variables, classes, types and procedures are always public.*

The visibility status of attributes is established in a class declaration, and later is inherited by all objects of this class.

Technically the visibility status of an attribute is in fact the visibility status of its reference, and the latter depends on the origin tag of the reference according to the following general rules:

### General visibility rules

1. A reference is visible in a state, if the origin tag of this reference
  - 1.1. either is \$ (global visibility), or
  - 1.2. coincides with the origin tag of the state (local visibility).
2. A reference must be visible whenever we intend to:
  - 2.1. get a value assigned to it in evaluating an expression,
  - 2.2. change the value assigned to it in executing an assignment instruction.
3. The origin tags of references and states are established when these references and states are created, and later they can't be changed.

We will say that a variable is *declared* in a state, if it is bound in the object of this state.

If a variable has been declared in a state, then we shall say that this state is a *hosting state* of this variable and of its value. If we declare an attribute in a class, then we say that this class is a *hosting class* of this attribute and of its value.

If an origin tag of an attribute is a name of a class, then this attribute is said to be *private for this class*. Such an attribute will be visible only in states whose origin tag is the name of the class. As we are going to see, the only such states will be the local state of the calls of procedures declared in this class.

### Operational visibility rules

Let `MyClass` be a class named 'MyClass', and let `myProc` be an arbitrary procedure declared in this class and named 'myProc'.

4. When we declare an attribute in a class called 'MyClass', then:
  - 4.1. if this attribute is to be private, then its origin tag is set to 'MyClass',
  - 4.2. if this attribute is to be public, then its origin tag is set to \$.
5. When we declare a variable in a state, then it is always public, and therefore the origin tag of its reference is \$. Note that a variable which is local to a procedure call is "locally public" for this call.
6. Local states of a call of `myProc` have origin tags 'MyClass', which means that private attributes of type 'MyClass' are visible in these states. Of course, public attributes are visible by principle.

7. When we pass an actual reference parameter to a procedure call, then its reference becomes the reference of the corresponding formal parameter (visibility is inherited).

Note that rule 6. allows for the construction of dedicated procedures, traditionally called *getters* and *setters*, to be used when we wish to reach private attributes of objects. This rule is “implemented” in the constructors of the denotations of value expressions (Sec. 6.4.1), and reference expressions (Sec. 6.4.3).

To go deeper into the details of these rules let’s analyze an example illustrated in Fig. 5.4-1. Consider a program that we shall call a *main program*, and assume that the execution of this program starts in a state whose origin tag is \$, which means that our program is not a body of a procedure. That is why we call it “main”.

Since, as we are going to see in Sec. 6.5, instructions may only change values assigned to attributes (but not their references), starting from the instruction of main program, all states will have a common fixed environment, let’s call it *global environment*, and a common fixed objecton, let call it *global objecton*. Together they will be components of all consecutive *global states* and their *global stores*. In our example the global state is the following:

- In the global environment it binds three items:
  - A class **MyClass** named ‘MyClass’ with:
    - a private class attribute ‘att1’ with origin tag ‘MyClass’,
    - a public class attribute ‘att2’ with origin tag \$,
    - a pre-procedure **myProc** named ‘myProc’,
    - an object pre-constructor named ‘myCons’.
  - A procedure **myProc** named ‘myProc’; its indicator is a pair of identifiers (‘MyClass’, ‘myProc’), but for the lack of space on the picture we show only one of them,
  - An object constructor named ‘myCons’; object constructors belong to the category of procedures (a comment as above),
- In the objecton it binds:
  - two (public) object variables ‘myObj1’, and ‘myObj2’ pointing to objects of type ‘MyClass’. As a rule, the objectons of these objects have been generated by an object constructor; note that each of them has one public attribute and one private one,
  - a public variable ‘var’.

Now, assume that one of the instructions of our main program is a call of the procedure **myProc** with one value parameter **MyObj1**, and one reference parameter **MyObj2**. This call creates a local state, and applies to it a program, that is the body of the procedure. The initial local state of this program includes (Sec. 6.6.3.2):

- a global environment inherited from the global state (we do not show it in the figure),
- a new local store with the origin tag that is the name of the class, where **myProc** was declared — in our case it is ‘MyClass’; the latter fact makes visible in this state all private attributes with origin tag ‘MyClass’.

According to the rules of passing actual parameters to formal parameters, described in Sec. 6.6.3.4, the objecton of the local store, let’s call it *local objecton*, binds two formal parameters. We assume additionally that the program of the body includes a declaration of a local variable ‘loc-var’. Altogether the local objecton binds, therefore, three public variables:

- a *local variable* ‘loc-var’ with origin tag \$, declared in the procedure body according to the rule 5.,
- a *formal value-parameter* ‘for-val-par’ pointing to a reference with a fresh token and the profile of *actual value-parameter* ‘myObj1’, that points to a twin of the value of ‘myObj1’,
- a *formal reference-parameter* ‘for-ref-par’ pointing to the reference of *actual reference-parameter* ‘myObj2’,

Now, let’s analyze the *visibility perspectives* of our main program that operates on the global state, and of the program that constitutes the body of our procedure, and operates on the local state:

**The visibility perspective of the main program:**

- The globally declared class `MyClass`, procedure `myProc` and object constructor `myCons`; note that in the global environment we have a procedure and a constructor, rather than a pre-procedure a pre-constructor (cf. Sec. 5.1 and Sec. 6.7.6).

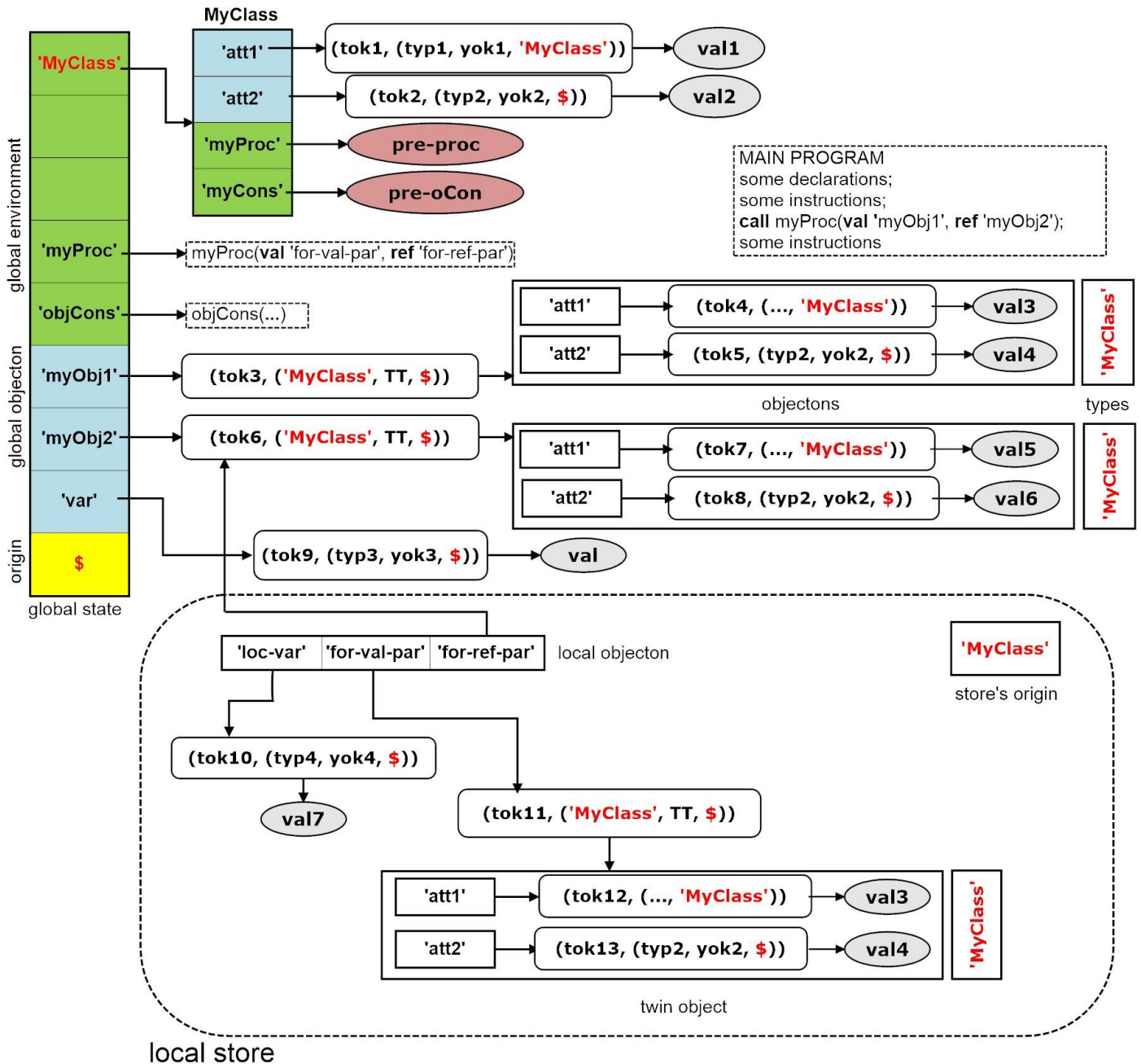


Fig. 5.4-1 An illustration of the visibility concept

- Two globally declared (public) object-variables `'myObj1'` and `'myObj2'`. We can assign their values to another (public) variable, we can assign a new value to this variable, and we can pass this variable as an actual parameter to a procedure call. However, we can't reach the private attribute `'att1'` of either of these objects, unless by a dedicated procedure (getter or setter) declared in `MyClass`. In our case this is `myProc`: the origin tag `'MyClass'` of the local store in its calls permits access to the private attribute `att1` of any object of class `MyClass`.
- One globally declared (public) variable `'var'`.

#### The visibility perspective of procedure body:

- All globally and locally declared classes and procedures.
- One local variable `'loc-var'` declared in the body program.

- One local object-variable ‘for-val-par’ that points to a twin of the value of actual parameter ‘myObj1’.
- One local object-variable ‘for-ref-par’ that points directly to the reference of ‘myObj2’ in the (inherited by the local store) global deposit.

Note that in both local object variables private attribute `att1` is visible since its origin tag is ‘MyClass’.

Let’s note at the end that when we call `locProc`, then the local state of `myProc` becomes for `locProc` a global state with all the consequences of this fact, but with one exception — its origin tag is ‘myClass’ rather than `$`. At the same time, however, all variables bound at this level, either locally declared, or passed as parameters, are public, and their visibility status is inherited by the local states of all levels of the locality.

## 6 DENOTATIONS

### 6.1 The carriers of the algebra of denotations

The denotations of our language will constitute an algebra of denotations **AlgDen** that will become a component of the diagram of algebras described in Sec. 3.4. The carriers of this algebra are the following:

#### Primitive carriers

|     |              |                                |                             |
|-----|--------------|--------------------------------|-----------------------------|
| ide | : Identifier | = ...                          | identifiers                 |
| prs | : PriSta     | = {'private', 'public'}        | privacy statuses indicators |
| loi | : ListOfIde  | = Identifier <sup>c*</sup>     | lists of identifiers        |
| cli | : ClaInd     | = {'empty-class'}   Identifier | class indicators            |

#### Applicative carriers<sup>37</sup>

|     |             |           |              |                                  |
|-----|-------------|-----------|--------------|----------------------------------|
| ved | : ValExpDen | = WfState | → ValueE     | value-expression denotations     |
| ted | : TypExpDen | = WfState | ↦ TypeE      | type-expression denotations      |
| red | : RefExpDen | = WfState | ↦ ReferenceE | reference-expression denotations |

#### Imperative carriers

|     |             |              |                     |  |
|-----|-------------|--------------|---------------------|--|
| dcd | : DecDen    | = WfState    | → WfState           | declaration denotations                        |
| pod | : ProOpeDen | = WfState    | ↦ WfState           | procedure opening denotation                   |
| ctd | : ClaTraDen | = Identifier | ↦ WfState → WfState | class-transformation denotations <sup>38</sup> |
| ind | : InsDen    | = WfState    | → WfState           | instruction denotations                        |
| ppd | : ProPreDen | = WfState    | → WfState           | program-preamble denotations                   |
| prd | : ProDen    | = WfState    | → WfState           | program denotations                            |

#### Declaration-oriented carriers

|     |             |                         |                              |
|-----|-------------|-------------------------|------------------------------|
| dse | : DecSec    | = ListOfIde x TypExpDen | declaration sections         |
| fpd | : ForParDen | = DecSec <sup>c*</sup>  | formal-parameter-denotations |
| apd | : ActParDen | = ListOfIde             | actual-parameter-denotations |

#### Signature carriers

|     |                |                          |  |
|-----|----------------|--------------------------|--|
| ips | : ImpProSigDen | = ForParDen x ForParDen  | imperative-procedure signature denotations |
| fps | : FunProSigDen | = ForParDen x TypExpDen  | functional-procedure signature denotations |
| ocs | : ObjConSigDen | = ForParDen x Identifier | object-constructor signature denotations   |

We tacitly assume that reserved words like 'private', 'public', 'empty class', etc. do not belong to Identifier.

The denotations of value expressions, declarations, instructions and programs are partial functions since all of them may generate infinite executions.

<sup>37</sup> In early programming languages used at the turn of the 1940s and 1950s, programs were sequences of simple instructions called "commands"; therefore, they could have been said to be written in an "imperative mood". Complex expressions came into play later in higher-order languages such as Algol and Fortran. Expressions were regarded as tools to be "applied" to get values. Over time, languages (practically) without instructions, i.e., where programs were expressions, started to emerge and were called "applicative languages". One of the first was Lisp — a language for manipulating lists.

<sup>38</sup> Why class transformer denotations take an identifier as an argument is explained in Sec. 6.7.3



Type expressions are used in the declarations of types, variables, class attributes and methods. The role of the domain of program preamble denotations will be explained in Sec. 6.3

Note that the domains of items — i.e., of values, references, types, procedure and classes — are not the carriers of our algebra, which means that they will not have their counterparts in the algebras of syntax. Values, references, types and classes will be (indirectly) represented by expressions, and procedures — by declarations and calls.

An applicative denotation is said to be *conservative*, if given a state that carries an error, returns this error as a result. A constructor of applicative denotations is said to be *diligent*, if given conservative denotations as arguments return a conservative denotation as a result. As we are going to see, all applicative denotations reachable in our language will be conservative.

An imperative denotation is said to be *conservative*, if given a state that carries an error, returns the same state as a result. A constructor of imperative denotations is said to be *diligent*, if given conservative denotations as arguments returns a conservative denotation as a result. Typical imperative denotations in **Lingua** will be conservative which implementationally means that once an error message is raised during the execution of a program, the execution aborts and the error message is signaled. However, our model of errors allows for an introduction of error-handling mechanisms, where occurrences of errors trigger recovery actions. An example of a corresponding not-diligent constructor of instruction denotations is discussed in Sec. 6.5.3.

In the subsections that follow we shall define the constructors of our algebra of denotations.

## 6.2 Identifiers, class indicators and privacy statuses

Identifiers, class indicators and privacy statuses have a singular character in our model since they are common for the algebras of denotations and syntax. We decided (a mathematical decision) that talking about the “denotations of identifiers” on one hand, and of “syntax of identifiers” on the other, and similarly for two remaining categories, would be a too-far going fundamentalism. We assume, therefore, that their denotational carriers and their abstract-, concrete- and colloquial syntactic carriers are the same, and are just sets of strings of characters.

Algebraically identifiers are built by zero-argument constructors, one for every identifier. Formally, each of such constructors is built by a metaconstructor **build-id** (a constructor of constructors) that given an identifier *ide* returns a constructor that given an empty tuples returns *ide*. Formally:

**build-id** : Identifier  $\mapsto$  (  $\mapsto$  Identifier )

**build-id.ide.()** = *ide*

for every *ide* : Identifier

where () denotes an empty tuple of arguments.

We recall in this place (cf. Sec. 2.13) that the future algebra of syntax of our language will be constructed as a homomorphic co-image of the (unique) reachable subalgebra of **AlgDen**. Consequently, only reachable denotations will have their counterparts at the level of syntax. Class indicators will be generated by two constructors:

**create-class-ind-of-empty** :  $\mapsto$  ClaInd

**create-class-ind-of-empty.()** = ‘empty-class’

**create-class-ind-of-parent** : Identifier  $\mapsto$  ClaInd

**create-class-ind-of-parent.ide** = *ide*

Their role will be explained in Sec. 6.7.3. Privacy statuses are also built by two constructors:

**build-ps-private** :  $\mapsto$  PriSta

**build-ps-private.()** = ‘private’

**build-ps-public** :  $\mapsto$  PriSta

**build-ps-public.()** = ‘public’

The role of these elements was explained in Sec. 5.4.



## 6.3 Programs and their segments

Before we proceed to the denotations of expressions, instructions and declaration we shall take a “strategic” decision about the future structure of our programs. We presume that they will consist of three *segments* sequentially composed in this order:

1. a *preamble* consisting of (sequentially interleaving) instructions and declarations,
2. one universal procedure-opening function **open-pro-den**; it is implicit in this rule that opening function can’t appear in preamble,
3. an instruction.

Of course, all mentioned above instructions may be structured, i.e. including other instructions. Besides, the first and the third segment may be trivial skip-segments.

As we will see, the procedure-opening command **open procedures** is a special tool for the elaboration of recursive procedures whose declarations may belong to different classes (details in Sec. 6.6.1 and 6.7.6).

The assumed restriction of the structure of programs has partly engineering and partly mathematical justification.

At the engineering side it should help programmers to better understand and control the “behaviors” of their programs. Declarations build tools to be used in programs, and therefore it seems reasonable to start from them in developing a program. In turn, instructions included in preambles are necessary for building values, and in particular objects, to initialize declared variables and attributes .

At the mathematical level our assumption will simplify the mechanism of passing and returning the references of formal reference-parameters of procedure calls (Sec. 6.6.3.5) and consequently also the rule of building correct procedure calls (Sec. 9.4.6.3). It also standardizes the process of correct program development (Sec. 9.4.2).

To formally express our assumption about the three segments of programs we introduce the following constructor:

$$\begin{aligned} \text{make-prog-den} &: \text{ProPreDen} \times \{\text{open-pro-den}\} \times \text{InsDen} \mapsto \text{ProDen} \\ \text{make-prog-den}(\text{ppd}, \text{pod}, \text{ind}) &= \text{ppd} \bullet \text{open-pro-den} \bullet \text{ind} \end{aligned}$$

where

$$\text{open-pro-den} : \text{WfState} \rightarrow \text{WfState}$$

is defined in Sec. 6.7.6 and describes a mechanism of a recursive elaboration in one step of all procedures declared in all classes. We also define three constructors of the declarations of program preambles:

$$\begin{aligned} \text{make-ppd-of-dcd} &: \text{DecDen} \mapsto \text{ProPreDen} && \text{an insertion} \\ \text{make-ppd-of-dcd.dcd} &= \text{dcd} \\ \text{make-ppd-of-ind} &: \text{InsDen} \mapsto \text{ProPreDen} && \text{an insertion} \\ \text{make-ppd-of-ind.ind} &= \text{ins} \\ \text{compose-ppd} &: \text{ProPreDen} \times \text{ProPreDen} \mapsto \text{ProPreDen} \\ \text{compose-ppd}(\text{ppd-1}, \text{ppd-2}) &= \text{ppd-1} \bullet \text{ppd-2} \end{aligned}$$

Set-theoretically first two constructors are identity functions, but algebraically they “make” program preambles out of declarations and instructions.

## 6.4 Expressions

### 6.4.1 Value expressions

The denotations of *value expressions* are partial functions, that given a state return a value or an error:

$$\text{ved} : \text{ValExpDen} = \text{WfState} \rightarrow \text{ValueE} \quad \text{value-expression denotations}$$

We split value expressions into six categories. All these categories belong to the common carrier of value expressions:

1. fixed-value expressions that return a typed data independently of the current state,
2. selection expressions that return a value pointed by a variable or an attribute of an object,
3. functional-procedure calls that return values built by procedures,
4. composed expressions associated with typed-data constructors.
5. boolean expressions,
6. conditional expressions.

The signatures of constructors of the denotations of value expressions are listed below.

#### Constructors of fixed-value-expression denotations

|                   |   |  |
|-------------------|---|--|
| <b>ved-bo.bo</b>  | : | $\mapsto$ ValExpDen for boo : {tt, ff} |
| <b>ved-in.int</b> | : | $\mapsto$ ValExpDen for int : Integer  |
| <b>ved-re.rea</b> | : | $\mapsto$ ValExpDen for rea : Real     |
| <b>ved-tx.tex</b> | : | $\mapsto$ ValExpDen for tex : Text     |

#### Constructors of selection-expression denotations

|               |                          |                     |
|---------------|--------------------------|---------------------|
| ved-variable  | : Identifier             | $\mapsto$ ValExpDen |
| ved-attribute | : ValExpDen x Identifier | $\mapsto$ ValExpDen |

#### Constructor of functional procedure calls

|                  |                                       |                     |
|------------------|---------------------------------------|---------------------|
| ved-call-fun-pro | : Identifier x Identifier x ActParDen | $\mapsto$ ValExpDen |
|------------------|---------------------------------------|---------------------|

#### Constructors based on typed-data constructors (examples)

|                 |                          |                     |
|-----------------|--------------------------|---------------------|
| ved-divide-re   | : ValExpDen x ValExpDen  | $\mapsto$ ValExpDen |
| ved-create-li   | : ValExpDen              | $\mapsto$ ValExpDen |
| ved-get-from-ar | : ValExpDen x ValExpDen  | $\mapsto$ ValExpDen |
| ved-get-from-rc | : ValExpDen x Identifier | $\mapsto$ ValExpDen |
| ...             |                          |                     |

#### Constructors of boolean-expression denotations

|           |                         |                     |
|-----------|-------------------------|---------------------|
| ved-equal | : ValExpDen x ValExpDen | $\mapsto$ ValExpDen |
| ved-less  | : ValExpDen x ValExpDen | $\mapsto$ ValExpDen |
| ved-and   | : ValExpDen x ValExpDen | $\mapsto$ ValExpDen |
| ved-or    | : ValExpDen x ValExpDen | $\mapsto$ ValExpDen |
| ved-not   | : ValExpDen             | $\mapsto$ ValExpDen |

#### Conditional-expression constructor

|        |                                     |                     |
|--------|-------------------------------------|---------------------|
| ved-if | : ValExpDen x ValExpDen x ValExpDen | $\mapsto$ ValExpDen |
|--------|-------------------------------------|---------------------|

Constructors of the first group are similar (in their nature) to constructors of identifiers described in Sec. 6.2. E.g., **ved-int** given an integer 3 returns a zero-argument constructor of value-expression denotations:

|                        |  |      |
|------------------------|--|------|
| <b>ved-int.3</b>       | : $\mapsto$ ValExpDen                    | i.e. |
| <b>ved-int.3</b>       | : $\mapsto$ WfState $\rightarrow$ ValueE |      |
| <b>ved-int.3().sta</b> | =  |      |
| is-error.sta           | $\rightarrow$ error.sta                  |      |
| <b>true</b>            | $\rightarrow$ (3, 'integer')             |      |

In this way, for every integer acceptable in our model we assume to have a dedicated constructor. Consequently, on the side of concrete syntax we shall be able to write constant-value expressions like 3, 245, or 340987502. If we had introduced only one zero-argument constructors corresponding to, say integer 1, then instead of writing 3 we had to write, e.g., ((1+1)+1).

As we are going to see, fixed-value expressions always evaluate to typed data or errors. The same will be true for composed expressions based on typed-data constructors. Consequently the only expressions that evaluate to objects will be selectors and functional-procedure calls. In the latter case procedure calls will return objects previously saved in stores and (possibly) modified by assigning new values to their attributes. Unlike in the case of typed-data expressions that return records, and may add or remove record attributes, object-values functional procedures may only modify the values assigned to attributes. That is an engineering decision.

The modifications of earlier created and declared objects are performed exclusively by instructions, and can only change values assigned to attributes. The only context where we can add a new attribute to an object are class declarations where we build class objects (Sec. 6.7.4.2). These objects are later used as patterns to build objects when we create object variables in stores by object constructors (Sec. 6.7).

The constructor that follows builds the denotations of expressions that return values assigned to state attributes, i.e. to variables:

ved-variable : Identifier  $\mapsto$  ValExpDen i.e.  
ved-variable : Identifier  $\mapsto$  WfState  $\rightarrow$  ValueE  
ved-variable.ide.sta  
is-error.sta  $\rightarrow$  error.sta  
**let**  
(env, (obn, dep, st-ota, sft, 'OK')) = sta  
obn.ide = ?  $\rightarrow$  'variable not declared'  
dep.(obn.ide) = ?  $\rightarrow$  'variable not initialized'  
**true**  $\rightarrow$  dep.(obn.ide)

The evaluation of a variable expression returns an error, if the variable hasn't been initialized<sup>39</sup>.

Next constructor corresponds to an expression that returns a value assigned to an attribute of an indicated object:

ved-attribute : ValExpDen x Identifier  $\mapsto$  ValExpDen i.e.  
ved-attribute : ValExpDen x Identifier  $\mapsto$  WfState  $\rightarrow$  ValueE  
ved-attribute.(ved, at-ide).sta =  
is-error.sta  $\rightarrow$  error.sta  
ved.sta = ??  $\rightarrow$  ??  
ved.sta : Error  $\rightarrow$  ved.sta  
ved.sta /: Object  $\rightarrow$  'object expected'  
**let**  
(ob-obn, cl-ide) = ved.sta  
ob-obn.at-ide = ?  $\rightarrow$  'attribute unknown'  
**let**  
(tok, (typ, ota)) = obn.ide the reference of ide in obn  
(env, (obn, dep, st-ota, sft, 'OK')) = sta  
dep.(ob-obn.at-ide) = ?  $\rightarrow$  'attribute not initialized'  
ota  $\neq$  \$ **and** ota  $\neq$  st-ota  $\rightarrow$  'attribute not visible'  
**true**  $\rightarrow$  dep.(ob-obn.at-ide)

An expression that returns a value assigned to an attribute of an object may be evaluated successfully only if this attribute is visible in the current state. Here we realize the rules 1 and 2.1 of Sec. 5.4.3. Next constructor corresponds to calling a functional procedure:

call-fun-pro : Identifier x Identifier x ActParDen  $\mapsto$  ValExpDen

<sup>39</sup> In their colloquial English programmers frequently say "a variable abc" when they mean (should say), "a variable expression abc". The difference between these concepts is clear at the level of abstract syntax where **build-id.abc.()** is a variable and **ved-variable.abc** is an expression. Then, at the level of colloquial syntax both are written as **abc**. which may lead to confusion.

We postpone its definition till Sec.6.6.4.2 where we discuss procedure calls.

As an example of a constructor based on a data constructor we show a constructor associated with the division of real numbers:

ved-divide-re: ValExpDen x ValExpDen  $\mapsto$  ValExpDen i.e.  
ved-divide-re: ValExpDen x ValExpDen  $\mapsto$  WfState  $\rightarrow$  ValueE  
ved-divide-re.(ved-1, ved-2).sta =  
  is-error.sta  $\rightarrow$  error.sta  
  ved-i.sta = ?  $\rightarrow$  ? for i = 1,2  
  ved-i.sta : Error  $\rightarrow$  ved-i.sta for i = 1,2  
  **let**  
    val-i = ved-i.sta for i = 1,2  
    val = td-divide-re.(val-1, val-2)  
  **true**  $\rightarrow$  val

Our constructor “calls” the typed-data constructor **td-divide-re** (see Sec. 4.3) that performs the following actions:

1. checks if **val-i**’s are of real types, and if **val-2** is different from zero,
2. divides data parts of these values; this constructor also checks if the result is not too large, and if it is so, it generates an error message indicating an overflow,
3. returns the computed quotient or an error.

Since in the domain of typed data (Sec. 4.3) we have not defined constructors of boolean data (explanation below), we have to define constructors of boolean-expression denotations now, and we define them “from scratch”. There are two groups of boolean expressions that we shall discuss. First group is built over comparison relations such as, e.g., an equality relation.

ved-equal : ValExpDen x ValExpDen  $\mapsto$  ValExpDen i.e.  
ved-equal : ValExpDen x ValExpDen  $\mapsto$  WfState  $\rightarrow$  ValueE  
ved-equal.(ved-1, ved-2).sta =  
  is-error.sta  $\rightarrow$  error.sta  
  ved-i.sta = ?  $\rightarrow$  ? for i = 1,2  
  ved-i.sta : Error  $\rightarrow$  ved-i.sta for i = 1,2  
  **let**  
    (cor-i, typ-i) = ved-i.sta for i = 1,2  
  typ-1  $\neq$  typ-2  $\rightarrow$  ‘compared values must be of the same type’  
  **not** comparable.typ-1  $\rightarrow$  ‘values not comparable’  
  cor-1 = cor-2  $\rightarrow$  (tt, ‘boolean’)  
  **true**  $\rightarrow$  (ff, ‘boolean’)

The metapredicate **comparable** is a parameter of our model and has been introduced in Sec. 4.2 to distinguish between comparable and not comparable values depending on their types. At this moment we do not decide which concrete categories (types) of values will be comparable, leaving it to an engineering decision of a language designer.

Second group of boolean constructors is associated with logical connectives. Below we show an example of such a constructor associated with alternative:

ved-or : ValExpDen x ValExpDen  $\mapsto$  ValExpDen i.e.  
ved-or : ValExpDen x ValExpDen  $\mapsto$  WfState  $\rightarrow$  ValueE  
ved-or.(ved-1, ved-2).sta =  
  is-error.sta  $\rightarrow$  error.sta  
  ved-1.sta = ?  $\rightarrow$  ?  
  ved-1.sta : Error  $\rightarrow$  ved-1.sta  
  **let**  
    (cor-1, typ-1) = ved-1.sta  
  typ-1  $\neq$  ‘boolean’  $\rightarrow$  ‘boolean value expected’

```

cor-1 = tt           → (tt, 'boolean')
ved-2.sta = ?        → ?
ved-2.sta : Error    → ved-2.sta
let
  (cor-2, typ-2) = ved-2.sta
  typ-2 ≠ 'boolean'  → 'boolean value expected'
  cor-2 = tt         → (tt, 'boolean')
  true              → (ff, 'boolean')

```

Note that the constructed expression denotation is not transparent for errors, and even not for undefinedness. If `ved-1` evaluates to `(tt, 'boolean')`, then the final result is `(tt, 'boolean')` even if the evaluation of `ved-2` generates a loop or an error, e.g. if `ved-2.sta` is not a boolean value. This evaluation pattern is referred to as *lazy evaluation*. The opposite is an *eager evaluation* — as in all remaining examples of constructors — where we evaluate both subexpressions in the first place, and only then try to calculate the final result. Due to the laziness of our constructor an expression like

$x > 0$  **implies**  $1/x > 0$  i.e.  $x \leq 0$  **or**  $1/x > 0$

is true for all values of  $x$  that are less or equal zero. Note that with an eager evaluation it would generate an error for  $x = 0$ . Note, however, that at the same time a “nonsensical” expression

$x > 0$  **implies**  $x+y$

is also true for  $x$  that are less or equal zero. In this case for  $x$  greater than zero our expression will generate an error message ‘boolean value expected’.

Our last constructor correspond to if-then-else-fi expressions.

`ved-if : ValExpDen x ValExpDen x ValExpDen  $\mapsto$  ValExpDen`

```

ved-if.(ved-1, ved-2, ved-3).sta =
  is-error.sta      → error.sta
  ved-1.sta = ?      → ?
  let
    val-1 = ved-1.sta
    val-1 : Error    → val-1
    let
      (cor-1, typ-1) = val-1
      typ-1 ≠ 'boolean' → 'boolean-expected'
      cor-1 = tt       → ved-2.sta
      cor-1 = ff       → ved-3.sta

```

Here we also have to do with a lazy evaluation. In this case the advantage of laziness is even better visible. Consider the following example written in an anticipated syntax:

**if**  $x > 0$  **then** `sqr(x)` **else** `sqr(-x)` **fi**

where `sqr(x)` denotes the square root of  $x$ . With an eager evaluation this expression evaluates to an error for all  $x$  except  $x = 0$ .

One methodological remark is necessary at the end and it concerns a question why we do not introduce in our algebra of denotations a carrier of boolean expressions, i.e., expressions with boolean values? We might use such expressions in building **if-then-else-fi** and **while-do-od** instructions, syntactically eliminating in this way the source of a typing errors, when a control expression generates a not-boolean value. Let us then analyze consequences of such a solution.

If we assume to have boolean expressions, i.e., a domain of their denotations `BooExpDen`, then it seems natural to have among them boolean variables. The denotations of such variables would be then constructed by the following constructor:

`boo-variable : Identifier  $\mapsto$  BooExpDen`

```

boo-variable : Identifier  $\mapsto$  WfState  $\rightarrow$  BooValE
boo-variable.ide.sta
  is-error.sta  $\rightarrow$  error.sta
  let
    (env, (obn, dep, st-ota, sft, 'OK')) = sta
    obn.ide = ?  $\rightarrow$  'variable not declared'
    dep.(obn.ide) = ?  $\rightarrow$  'variable not initialized'
    sort-va.(dep.(obn.ide))  $\neq$  'boolean'  $\rightarrow$  'boolean value expected'
    true  $\rightarrow$  dep.(obn.ide)

```

This constructor checks if the value of the variable is boolean, whereas the earlier defined constructor

```
val-variable : Identifier  $\mapsto$  BooExpDen
```

does not check this property. Consequently at the level of abstract syntax (Sec. 7.2.3) we would have two categories of variables

```

ved-variable(ide) and
boo-variable(ide)

```

So far, everything is fine. But what about concrete syntaxes of variables? Can the abstract-to-concrete homomorphism glue them together? The answer is, not, since their denotations are different. The consequence of having boolean expressions as a syntactic category is therefore to have two syntactic categories of variables. Such a solution, although technically possible, seems not quite practical, since in such a case variables had to be somehow marked.

## 6.4.2 Type expressions

The denotations of type expressions are total functions that given a state return a type or an error:

```
ted : TypExpDen = WfState  $\mapsto$  TypeE
```

Type expressions are used in four contexts:

1. in type declarations, where we build a new type and store it in a type environment of a class for future use,
2. in variable declarations, where we declare a new variable and assign a profile to it (we recall that types are components of profiles),
3. in attribute declaration — analogously,
4. in pre-procedure declarations, where we assign profiles to formal parameters.

The signatures of constructors of type-expressions denotations are the following:

```

ted-create-bo   :  $\mapsto$  TypExpDen
ted-create-in   :  $\mapsto$  TypExpDen
ted-create-re   :  $\mapsto$  TypExpDen
ted-create-tx   :  $\mapsto$  TypExpDen
ted-create-ot   : Identifier  $\mapsto$  TypExpDen

ted-constant    : Identifier x Identifier  $\mapsto$  TypExpDen
ted-create-li    : TypExpDen  $\mapsto$  TypExpDen
ted-create-ar    : TypExpDen x ValExpDen  $\rightarrow$  TypExpDen
ted-create-rc    : Identifier x TypExpDen  $\mapsto$  TypExpDen
ted-put-to-rc    : Identifier x TypExpDen x TypExpDen  $\mapsto$  TypExpDen

```

First constructor is a zero-argument constructor that creates a boolean-type expression denotation “out of nothing”:

```

ted-create-bo().sta =
  is-error.sta  $\rightarrow$  error.sta

```

**true**             $\rightarrow$  'boolean'

The presence and the role of this constructor in **AlgDen** is the same as in the case of the constructors of identifiers (cf. Sec.6.2). The remaining zero-argument constructors are defined in a similar way.

Next constructor creates an object type out of an identifier. Algebraically it is an insertion, hence:

```
ted-create-ot.ide.sta =
  is-error.sta   $\rightarrow$  error.sta
  true          $\rightarrow$  ide
```

Although intentionally **ide** is supposed to be the name of a class, we do not check it here, since — as we will see in Sec. 6.7.4.2 — we may need to define an object type (temporarily) associated with a class which “hasn’t been declared yet”. However, as we will see in Sec. 6.7.4.3, such “undefined object types”, will not be declarable in type environments).

Constructor **ted-constant** corresponds to a type-constant. This constructor describes the action of reading a previously declared type from a type environment of a class:

```
ted-constant.(ide-cl, ide-ty).sta =
  is-error.sta       $\rightarrow$  error.sta
  let
    ((cle, mee, cov), sto) = sta
    cle. ide-cl = ?    $\rightarrow$  'class unknown'
  let
    (ide-cl, tye, mee, obn) = cle. ide-cl
    tye. ide-ty = ?     $\rightarrow$  'type unknown'
    tye. ide-ty =  $\emptyset$   $\rightarrow$  'type not concretized'
  true               $\rightarrow$  tye. ide-ty
```

well-formedness of sta

We are talking here about “constants”, since a type once assigned to an identifier in a class can’t be changed.

Remaining constructors refer to (call) type constructors from the algebra of types. For instance:

```
ted-create-ar.(ide, ted, ved).sta =
  is-error.sta       $\rightarrow$  error.sta
  ted.sta : Error     $\rightarrow$  ted.sta
  ved.sta = ?         $\rightarrow$  ?
  ved.sta : Error     $\rightarrow$  ved.sta
  let
    a-typ            = ted.sta
    (dat, d-typ)      = ved.sta
    d-typ  $\neq$  'integer'  $\rightarrow$  'integer expected'
  true               $\rightarrow$  ty-create-rc.(ide, a-typ, dat)
```

Note that the data-type constructor **ty-create-rc** checks if **dat** is positive and not too large (cf. Sec. 4.2).

### 6.4.3 Reference expressions

Reference expressions in our model are used to indicate references where we assign values in executing assignment instructions. In programming languages without deep value-structures as our objects, a reference on the left-hand side of an assignment is represented by a single identifier. In our case the situation is different, since we may wish to assign a value to a deep reference in an object, as, e.g.,

**node.next.no** := 11

(cf. example discussed in Sec. 5.1). To handle this, we introduce expressions that given a state return a reference, or an error:

**red** : RefExpDen = WfState  $\mapsto$  ReferenceE



We shall need only two constructors of the denotations of such expressions. The first one corresponds to a single variable:

ref-variable : Identifier  $\mapsto$  RefExpDen i.e.  
 ref-variable : Identifier  $\mapsto$  WfState  $\mapsto$  ReferenceE  
 ref-variable.ide.sta =  
   is-error.sta  $\rightarrow$  error.sta  
   **let**  
     (env, (obn, dep, st-ota, sft, 'OK')) = sta  
     obn.ide = ?  $\rightarrow$  'variable not declared'  
   **true**  $\rightarrow$  obn.ide

Our second constructor builds expression denotations that may return references assigned to deep attributes of objects:

ref-attribute : ValExpDen x Identifier  $\mapsto$  RefExpDen i.e.  
 ref-attribute : ValExpDen x Identifier  $\mapsto$  WfState  $\mapsto$  ReferenceE  
 ref-attribute.(ved, at-ide).sta =  
   is-error.sta  $\rightarrow$  error.sta  
   ved.sta = ?  $\rightarrow$  ?  
   ved.sta : Error  $\rightarrow$  ved.sta  
   ved.sta /: Object  $\rightarrow$  'object expected'  
   **let**  
     (va-obn, va-ide) = ved.sta  
     (env, (obn, dep, st-ota, sft, 'OK')) = sta  
     va-obn.at-ide = ?  $\rightarrow$  'attribute unknown'  
   **let**  
     (tok, (typ, at-ota)) = va-obn.at-ide  
     at-ota  $\neq$  \$ **and** at-ota  $\neq$  st-ota  $\rightarrow$  'attribute not visible'  
   **true**  $\rightarrow$  va-obn.at-ide

Here we realize the rules 1. and 2.2 of Sec. 5.4.3. As we see, references computed by reference expressions are always pointed either by variables or by object attributes. In other words, the only operations that allow us to get references are selection operations. It is, of course, an engineering decision.

## 6.5 Instructions

### 6.5.1 Signatures of constructors

Instructions modify states by assigning new values to variables and attributes. They may also introduce an error message to the error register of a state (see Sec.5.3). We shall define the following constructors of instruction denotations:

#### atomic instructions

|              |  |                  |                     |
|--------------|--|------------------|---------------------|
| assign       | : RefExpDen x ValExpDen                            | $\mapsto$ InsDen | assignments         |
| call-imp-pro | : Identifier x Identifier x ActParDen x ActParDen  | $\mapsto$ InsDen | imp. proc. calls    |
| call-obj-con | : Identifier x Identifier x Identifier x ActParDen | $\mapsto$ InsDen | obj. const. calls   |
| skip-ins     | :  | $\mapsto$ InsDen | trivial instruction |

#### structural instructions

|             |                               |                  |                          |
|-------------|-------------------------------|------------------|--------------------------|
| if          | : ValExpDen x InsDen x InsDen | $\mapsto$ InsDen | conditional instructions |
| if-error    | : ValExpDen x InsDen          | $\mapsto$ InsDen | error elaboration        |
| while       | : ValExpDen x InsDen          | $\mapsto$ InsDen | while loops              |
| compose-ins | : InsDen x InsDen             | $\mapsto$ InsDen | sequential compos.       |

Atomic instructions are called in this way, since they do not include other instructions as their components. The skip-instruction constructor is defined as follows:

$\text{skip-ins}().\text{sta} = \text{sta}$

It has a technical character and has been introduced to cover the case of functional procedures (Sec.6.6.4.1) whose bodies consist of an expression alone, i.e. without a preceding program, or if we want to define if-then-fi as if-then-else skip fi.

The calls of imperative procedures and of object constructors will be described in Sec. 6.6.3.6 and Sec. 6.6.5.3.

## 6.5.2 Assignment instructions

An *assignment instruction* computes a reference and a value, and then assigns this value to this reference in the current deposit:

```

assign : RefExpDen x ValExpDen  $\mapsto$  InsDen
assign : RefExpDen x ValExpDen  $\mapsto$  WfState  $\rightarrow$  WfState
assign.(red, ved).sta =
  is-error.sta            $\rightarrow$  sta
  ved.sta = ?             $\rightarrow$  ?
  ved.sta : Error         $\rightarrow$  sta  $\blacktriangleleft$  ved.sta
  red.sta : Error         $\rightarrow$  sta  $\blacktriangleleft$  red.sta
let
  val                    = ved.sta
  ref                    = red.sta
  (tok, (typ, re-ota))    = ref
  ((cle, pre, cov), (obn, dep, st-ota, sft, 'OK')) = sta
  re-ota  $\neq$  $ and re-ota  $\neq$  st-ota  $\rightarrow$  sta  $\blacktriangleleft$  'reference not visible'
  not ref VRA.cov val       $\rightarrow$  sta  $\blacktriangleleft$  'incompatibility of types'
let
  new-sta = (env, (obn, dep[ref/val], st-ota, sft, 'OK'))
true       $\rightarrow$  new-sta

```

In this definition we realize the rule 2.2 of Sec. 5.4.3.

## 6.5.3 Structural instructions

*Structural instructions* are built from atomic instructions using four constructors announced in Sec. 6.5.1. A conditional composition of instructions is defined as follows:

```

if : ValExpDen x InsDen x InsDen  $\mapsto$  InsDen
if.(ved, ind-1, ind-2).sta =
  is-error.sta            $\rightarrow$  sta
  ved.sta = ?             $\rightarrow$  ?
  ved.sta : Error         $\rightarrow$  sta  $\blacktriangleleft$  ved.sta
let
  val = ved.sta
  val : Object       $\rightarrow$  sta  $\blacktriangleleft$  'typed data expected'
let
  (dat, typ) = val
  typ  $\neq$  'boolean'  $\rightarrow$  sta  $\blacktriangleleft$  'boolean value expected'
  dat = tt           $\rightarrow$  ind-1.sta
true               $\rightarrow$  ind-2.sta

```

Note that due to while loops (see below) and procedure calls (Sec. 6.6) the execution of both component instructions may be infinite, which means that the states ind-1.sta and ind-2.sta may be undefined.

The next structural constructor is related to an *error-handling mechanism*. It activates a *rescue action* that is an instruction associated with an error message indicated by value expression, called *error trap*, whose value is a word identical with this message.

$\text{if-error} : \text{ValExpDen} \times \text{InsDen} \mapsto \text{InsDen}$

```

if-error.(ved, ind).sta =
  not is-error.sta → sta
  let
    message = error.sta
    sta-1    = sta ◀ 'OK'
  ved.sta-1 = ? → ?
  let
    val = ved.sta-1
  val : Error → sta ◀ 'trap generates an error'
  let
    (cor, typ) = val
  typ ≠ 'text' → sta ◀ 'text expected'
  cor ≠ message → sta ◀ 'trap not adequate'
  ind.sta-1 = ? → ?
  let
    sta-2 = ind.sta-1
  is-error.sta-2 → sta ◀ 'rescue action generates an error'
  true → sta-2

```

If the input-state *sta* does not carry an error, then this state becomes the output state, since there is no error to handle.

In the opposite case, a temporary state *sta-1* is created by removing error *message* from *sta*. In the new state, we compute the value of the trap expression *ved*. Seven situations may happen in this moment:

1. the evaluation of trap expression does not terminate,
2. the evaluation terminates, but the computed value is an error,
3. the computed value is not a text value,
4. the computed value is a text value, but its data part is different from the error message that we want to trap,
5. the computed value carries the trapped message, but the rescue instruction does not terminate ,
6. the rescue instruction terminates but it generates an error message itself,
7. the rescue instruction terminates without an error, and its terminal state is the resulting state.

Of course, the above constructor should be regarded as an example, only showing that error-handling mechanisms may be described in our model.

The definition of the constructor of **while** loops involves a fixed-point definition of the constructed instruction:

$\text{while} : \text{ValExpDen} \times \text{InsDen} \mapsto \text{InsDen}$

```

while.(ved, ind).sta =
  is-error.sta → sta
  ved.sta = ? → ?
  ved.sta : Error → sta ◀ ved.sta
  let
    val = ved.sta
  val : Object → 'typed data expected'
  let
    (dat, typ) = val
  typ ≠ 'boolean' → sta ◀ 'boolean value expected'
  dat = ff → sta
  ind.sta = ? → ?

```

```

let
  sta-1 = ind.sta
  is-error.sta-1 → sta-1
  true → (while.(ved, ind)).sta-1

```

Notice that the unique (least) solution of this equation is not the **while** constructor, but the effect of its application to its arguments, i.e. a partial function  $\text{while}.\text{(ved, ind)} : \text{InsDen}$ .

Due to **while** instructions, the denotations of instructions may be partial functions. The partiality of  $\text{while}.\text{(ved, ind)}$  may happen in three situations:

1. the evaluation of the boolean expression **ved** does not terminate; this may be the case if **ved** calls a functional procedure,
2. the execution of the body **ind** does not terminate,
3. the execution of the “main loop” does not terminate.

**Comment 6.5.3-1** In the definition of **while** we have to do with a fixed-point equation in a CPO of partial functions **InsDen** (Sec. 2.7). For any pair **(ved, ind)** the solution of this equation is the denotation:

**while.(ved, ind) : State → State**

To see this equation written explicitly in our CPO, let us introduce the following notations:

```

NotOK   = {(sta, sta) | is-error.sta}
ExpEr   = {(sta, sta ◀ ved.sta) | ved.sta : Error}
IsObj   = {(sta, sta ◀ 'typed data expected') | ved.sta : Object}
NotBoo  = {(sta, sta ◀ 'boolean-expected') | ved.sta is not a boolean value}
FF      = {(sta, sta) | ved.sta = (ff, 'boolean')}
TT      = {(sta, sta) | ved.sta = (tt, 'boolean')}

```

Now, our equation is the following:

**X = NotOK | ExpEr | IsObj | NotBoo | FF | TT • ind • X**

Since the operators  $|$  and  $\bullet$  are continuous, the least solution of that equation exists, and since the coefficients of that equations have mutually disjoint domains, from Theorem 2.7-1 we may conclude that its solution is a function, and may be described by the following formula:

**X = (TT • ind)\* • (NotOK | ExpEr | IsObj | NotBoo | FF)**

The last constructor of structural instructions corresponds to sequential composition of instructions, and is the following:

```

compose-ins : InsDen x InsDen ↦ InsDen
compose-ins.(ind-1, ind-2) = ind-1 • ind-2

```

Sequentially composed instructions are executed one after another.

## 6.6 Methods

### 6.6.1 An overview of methods

Methods in our model fall into three basic operational categories:

- imperative methods,
- object-constructor methods,
- functional methods.

In each of these categories a method may be *abstract* or *concrete*. Abstract methods will be otherwise called *procedure signatures*, and concrete methods — *procedures* or *object constructors* respectively. The domain of methods and related domains are defined by the following equations (for actual-parameter denotations and the denotations of signature see Sec. 6.6.2):

**met** : Method = Procedure | ProSigDen methods

|     |                |  |  |                                   |
|-----|----------------|--|--|-----------------------------------|
| pro | : Procedure    | = ImpPro   FunPro   ObjCon                   |  | procedures                        |
| ipr | : ImpPro       | = ActParDen x ActParDen                      | $\mapsto \text{Store} \rightarrow \text{Store}$  | imperative procedures             |
| fpr | : FunPro       | = ActParDen x TypExpDen                      | $\mapsto \text{Store} \rightarrow \text{ValueE}$ | functional procedures             |
| oco | : ObjCon       | = ActParDen x Identifier                     | $\mapsto \text{Store} \rightarrow \text{Store}$  | object constructors <sup>40</sup> |
| prs | : ImpProSigDen | = ForParDen x ForParDen                      |  |                                   |
| prs | : FunProSigDen | = ForParDen x TypExpDen                      |  |                                   |
| prs | : ObjConSigDen | = ForParDen x Identifier                     |  |                                   |
| prs | : ProSigDen    | = ImpProSigDen   FunProSigDen   ObjConSigDen |  | proc. sig. denotations            |

Note that imperative procedures take two lists of actual parameters, whereas functional procedures take only one. As we will see in Sec. 6.6.3, imperative procedures will have value and reference parameters, whereas functional procedures (Sec. 6.6.4) will have only value parameters. This is an engineering decision to ensure that functional procedures have no side effects. In the same spirit, we will assume that imperative procedures will have no global variables. Both mechanisms could have been easily described in a denotational model, but would considerably complicate the correctness issue of these procedures.

It must be emphasized that domains associated with concrete methods are not the carriers of our algebra of denotations (cf. Sec. 6.1), and consequently they will not have syntactic counterparts. This is why we are not talking about “procedure denotations”, but about “procedures” as such. At the side of syntax we will only have procedure declarations and calls, and their denotation will belong to the denotations of declarations and of instructions respectively.

In turn, procedure signatures will be represented at the side of syntax, and therefore we talk about their denotations. Their simple constructors will be defined in Sec. 6.6.2.

It may be worth mentioning in this place that procedures and procedure-signature denotations belong to two different worlds. Procedures are functions that given actual parameters return store-to-store functions. In turn, signatures are not functions, but lists of formal-parameter denotations. Signatures may be said to be “incomplete procedure-declarations” — they have no bodies (yet) — whereas procedures are the effects of open-procedure declarations (Sec. 6.7.6).

Procedures are functions which taking *actual parameters* return store-to-store or store-to-value functions. As we are going to see, in the case of imperative procedures reference parameters will constitute a unique “communication channel” between procedures and the “external word”. A possible alternative might be global variables, but such a solution would complicate construction rules of correct procedure calls (cf. Sec. 9.4.6.3), and besides is — in our opinion — error prone. We want to make sure that all interactions of procedures with global states are explicit in their declarations.

A second important issue about procedures is that they modify stores rather than states. Although procedure calls will take states as arguments, we can’t assume procedures to do the same, since in that case they would take, via states, other procedures as arguments — even themselves. In the case of imperative procedures such a situation would lead to (simplifying a little) the following domain equations:

$$\begin{aligned} \text{Procedure} &= \text{State} \rightarrow \text{State} \\ \text{State} &= \text{Identifier} \Rightarrow \text{Procedure} \end{aligned}$$

<sup>40</sup> “Object constructors” should not be confused with “constructors in algebras”. The former constitute a category of procedures, whereas the latter are functions “between” carriers of algebras. We decided to use the same word in both cases because the literature has already established customs to call them that way.

This set can't be solved on the ground of usual set theory, since no function can take itself as an argument. Self-applicable functions constitute so called *reflexive domains*<sup>41</sup>, and have been used in early denotational models of Algol 60, where a procedure can take itself as a procedural parameter<sup>42</sup>.

After these explanations, let's note that to use procedures in a programming language we need tools to:

- create them,
- declare them, i.e. to save them in states,
- call and execute them.

In earlier versions of our denotational model (cf. [30], [32], [35], [39]) procedures were created by declarations out of procedures' components and were saved in the environments of states. Recursive procedures were defined as the least solutions of single fixed-point equations and mutually recursive procedures as the least solutions of sets of such equations.

In our case, since we want to allow procedures declared in a class to call (possibly recursively) not only other procedures in this class but also procedures in other classes, we would have to simultaneously elaborate all procedure declarations of all classes, meaning that all classes would have to be declared simultaneously and recursively. In our opinion, such a solution would lead to too many technical complications.

Facing this problem we decided to move the creation of procedures from the time of their declarations in classes to a later time when all classes have been declared, but prior to the time when procedures are called. Technically the declarations of procedures in classes will not create and store procedures, but only *pre-procedures* that will be later used to create all procedures "in one step" once all classes have been declared. In this step programs perform one global declaration **open-pro-den** (Sec. 6.7.6). To simplify future rules of program construction we have assumed in Sec. 6.3 that this operation appears only once in every program and is located on the borderline between declarations and instructions.

As a consequence of our decision, pre-procedures will be defined as functions that given an environment return a procedure. The domains of pre-procedures are therefore the following:

|                                 |  |                           |
|---------------------------------|--|---------------------------|
| $\text{ppr} : \text{PrePro}$    | $= \text{ImpPrePro} \mid \text{FunPrePro} \mid \text{ObjPreCon}$ | pre-procedures            |
| $\text{ipp} : \text{ImpPrePro}$ | $= \text{Env} \mapsto \text{ImpPro}$                             | imperative pre-procedures |
| $\text{fpp} : \text{FunPrePro}$ | $= \text{Env} \mapsto \text{FunPro}$                             | functional pre-procedures |
| $\text{opc} : \text{ObjPreCon}$ | $= \text{Env} \mapsto \text{ObjCon}$                             | object pre-constructors   |

When a procedure **pro** is called, we execute the corresponding **pre-pro** in a declaration time environment **dt-env**, i.e., we execute the function

$\text{pre-pro.dt-env} : \text{ActParDen} \mapsto \text{Store} \rightarrow \text{Store}$

which given a call-time store returns a new store. The declaration-time environment is common to all procedures, and is the environment passed to **open-procedures**. Since, by our assumption, no declarations follow **open-procedures**, all states that follow the execution of this declaration have a common environment which differs from the declaration-time environment **dt-env** only by having procedures declared in procedure environment **pre**.

## 6.6.2 Signatures and parameters

We start from two simple constructors of lists (tuples) of identifiers:

|                            |  |
|----------------------------|--|
| $\text{build-empty-loi} :$ | $\mapsto \text{ListOfIde}$   |
| $\text{add-to-loi} :$      | $\text{Identifier} \times \text{ListOfIde} \mapsto \text{ListOfIde}$ |

<sup>41</sup> A model of self-applicable functions has been described by Dana Scott and Christopher Strachey [89] in 1970., but it is technical rather complicated. Independently it turned out that the use of self-applicable functions in programming may be error prone.

<sup>42</sup> This mechanism was implemented in Algol 60 by the so-called "copy rule", where a compiler or interpreter copied the text of a procedure body into the context of a program where this procedure was to be used.



We skip their obvious definitions. Given this domain we may define *declaration sections* that consist of lists of identifiers followed by a type-expression denotation:

$$\text{dse} : \text{DecSec} = \text{ListOfIde} \times \text{TypExpDen}$$

their (obvious) constructor is the following:

$$\text{build-dse} : \text{ListOfIde} \times \text{TypExpDen} \mapsto \text{DecSec}$$

Such a section expresses the fact that the given identifiers are formal parameters of a given type, e.g., at the side of syntax:

*x, y, z array-of-integers*

*Formal-parameter denotations* are tuples of declaration sections, hence we need two constructors to build their domain:

$$\begin{aligned} \text{build-empty-fpd} & : & \mapsto \text{ForParDen} \\ \text{add-to-fpd} & : \text{DecSec} \times \text{ForParDen} \mapsto \text{ForParDen} \end{aligned}$$

First constructor builds an empty section (empty tuple), the second — adds a new section to a parameter denotation. Their definitions are obvious. Since actual parameter denotations are just lists of identifiers, we need only one (insertion) constructor to build them:

$$\text{build-apd} : \text{ListOfIde} \mapsto \text{ActParDen}$$

Now, we can show signatures of three constructors of the domains of three categories of signatures:

$$\begin{aligned} \text{build-ipsd} & : \text{ForParDen} \times \text{ForParDen} \mapsto \text{ImpProSigDen} \\ \text{build-fpsd} & : \text{ForParDen} \times \text{TypExpDen} \mapsto \text{FunProSigDen} \\ \text{build-ocsd} & : \text{ForParDen} \times \text{Identifier} \mapsto \text{ObjConSigDen} \end{aligned}$$

Set-theoretically these functions are just identities, e.g.

$$\text{build-ipsd}(\text{fpd-v}, \text{fpd-r}) = (\text{fpd-v}, \text{fpd-r})$$

but they play a role in the algebra of denotations since they make a denotation of an imperative-procedure signature out of two list of formal parameter denotations.

At the end let us explain why we assume that lists of actual parameters are identifiers rather than value expressions or respectively reference expressions.

Assuming actual value parameters to be arbitrary value expressions would lead to significant technical problems, since such expressions may include procedure calls, and therefore, recursion might “bind” procedures not only via calls in bodies but also via actual parameters. A technical solution might be to introduce a separate category of value expressions without procedure calls, but this would again complicate the model. Besides, if we resign from recursion via parameters, then, instead of making an expression a parameter, we may assign its value to an identifier and use it later as a parameter.

Regarding actual reference parameters we might assume them to be reference expressions without too much of a “technical cost”, but still it complicates our model (that is already not very simple), and besides, instead of passing as a parameter a “deep reference” of an object, a programmer can pass a reference to this object and reach its deeper level by a reference expression in procedure body.

## 6.6.3 Imperative pre-procedures

### 6.6.3.1 An intuitive introduction

First step on the way to understand the mechanism of imperative procedures is to understand the execution of their calls. Since an imperative-procedure call is an instruction, it takes an initial state, and transforms it into a terminal state. These states will be called *global states* to distinguish them from *local states* that the procedure creates to execute its body.



The execution of a call of a procedure declared in class **MyClass** is performed in four steps illustrated in Fig. 6.6-1.

1. A *local initial store* **li-sto** is created where initially the only variables bound in the objecton are the identifiers of formal parameters. Value parameters point to new references, and these references point to the twins (Sec. 4.4) of the values of actual value-parameters. Reference parameters point to the references of actual parameters. Local initial deposit carries all reference of the global deposit, but all these references, except the references of reference parameters, are orphan references. The origin tag of the local store is **MyClass**.
2. A *local initial state* is created by combining a *declaration-time environment* **dt-env** with the local initial store. The declaration-time environment must be, therefore, somehow “remembered” when the procedure is declared.
3. The *local initial state* is transformed by the body of the procedure (a deep program) into a *local terminal state* (**lt-env**, **lt-sto**). During the execution of the body some local (temporary) value variables, classes, types and procedures may be declared.
4. The local terminal state is transformed into a *global terminal state*, with the (unchanged) *global initial environment* **gi-env** and a *global-terminal store* **gt-sto**, where actual reference parameters regain access to their earlier references. All locally declared items cease to exist.

One comment is necessary about the declaration-time environment mentioned in point 2. As was already said in Sec. 6.6.1, this environment is “loaded” to a procedure when it is created by a global declaration **open-procedures**. Since this declaration is the last one in every program (Sec. 6.3), all consecutive states of the program have a common environment which is the output environment of the global declaration **open-procedures**. Consequently, in writing procedure bodies, programmers may refer not only to items (procedures, types and covering relations) that have been declared earlier, but also to these that are going to be declared in the future. And by our rule of syntax, there are no declarations that follow **open-procedures** in a program.

#### declaration-time environment

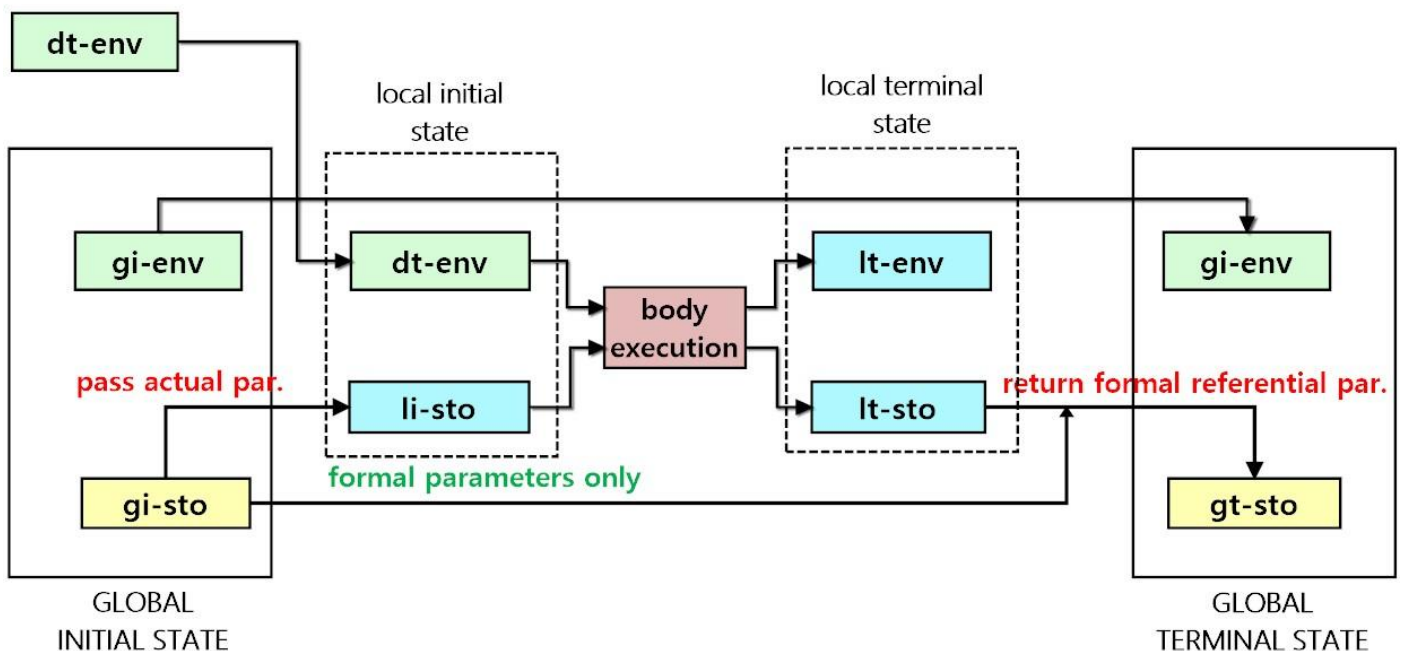


Fig. 6.6-1 The execution of an imperative-procedure call

#### 6.6.3.2 Creating imperative pre-procedures

A formal description of the execution illustrated in Fig. 6.6-1 is included in the definition of a constructor of imperative pre-procedures. We recall that pre-procedures became procedures when they receive an environment

as an argument. In the definition below we use two functions describing the mechanisms of passing and returning parameters, which will be defined later in Sec.6.6.3.4 and Sec.6.6.3.5 respectively.

```

create-imp-pre-pro : ImpProSigDen x ProDen x Identifier  $\mapsto$  ImpPrePro
create-imp-pre-pro : ForParDen x ForParDen x ProDen x Identifier  $\mapsto$ 
     $\mapsto$  Env  $\mapsto$  ActParDen x ActParDen  $\mapsto$  Store  $\rightarrow$  Store
create-imp-pre-pro.(fpd-v, fpd-r, prd, cl-ide).dt-env.(apd-v, apd-r).ct-sto =
  is-error.ct-sto  $\rightarrow$  ct-sto                                dt- declaration time
  let                                                    ct- call time
    li-sto = pass-actual.(fpd-v, fpd-r, apd-v, apd-r, cl-ide).dt-env.ct-sto    li- local initial
  is-error.li-sto  $\rightarrow$  ct-sto  $\blacktriangleleft$  error.li-sto
  let
    li-sta = (dt-env, li-sto)                                local initial state
  prd.li-sta = ?  $\rightarrow$  ?
  let
    lt-sta = prd.li-sta                                    local terminal state
  is-error.lt-sta  $\rightarrow$  ct-sto  $\blacktriangleleft$  error.lt-sta
  let
    (dt-cle, dt-pre, dt-cov) = dt-env
    (lt-env, lt-sto) = lt-sta
    gt-sto = return-formal.fpd-r.ct-sto.lt-sto.dt-cov
  is-error.gt-sto  $\rightarrow$  ct-sto  $\blacktriangleleft$  error.gt-sto
  true  $\rightarrow$  gt-sto                                global terminal store

```

First, the pre-procedure builds a *local initial store* by passing actual parameters to formal parameters and by setting **cl-ide**, the name of a class, as the origin tag of the store. As we will see in Sec. 6.7.4.6, **cl-ide** will be the class name where our pre-procedure will be defined. We will also see (Sec. 6.6.3.4) that the local initial store **li-sto** will include a local initial objecton and a global call-time store extended only by the references of formal value parameter.

Then, it creates a *local initial state* by combining the declaration-time environment with the call-time store.

The local initial state is transformed into a local terminal state by a program **prd** that constitutes the body of the procedure.

If this transformation terminates, and does not issue an error, then the global terminal store is created by returning the references of formal reference parameters to actual reference parameters, and by going back to the call-time origin tag. If this store carries no error message, then it is issued by the procedure. Next the mechanism of procedure call (Sec. 6.6.3.6 and Fig. 6.6-1) combines the global-terminal store with call-time environment into global-terminal state.

### 6.6.3.3 A static compatibility of parameters

The first step in passing actual parameters to formal parameters consist in checking if they are statically compatible with each other, i.e., if the lists of corresponding identifiers are of the same length, and additionally there are no repetitions of identifiers on the list of formal parameters. To formalize this checking process we define two auxiliary functions. We skip their (rather obvious, but tedious) formal definitions showing only examples:

```

list-of-for-par : ForParDen  $\mapsto$  (Identifier x TypExpDen)c*          e.g.
list-of-for-par.(((x, y, z), ted-1), ((q, r), ted-2)) = ((x, ted-1), (y, ted-1), (z, ted-1), (q, ted-2), (r, ted-2))

list-of-ide : (Identifier x TypExpDen)c*  $\mapsto$  Identifierc*          e.g.
list-of-ide.((x, ted-1), (y, ted-1), (z, ted-1), (q, ted-2), (r, ted-2)) = (x, y, z, q, r)

```

Now, the checking function is defined as follows:

```

statically-compatible : ForParDen x ForParDen x ActParDen x ActParDen  $\mapsto$  Error | {'OK'}
statically-compatible.(fpd-v, fpd-r, apd-v, apd-r) =

```

```

let
  for-val-par = list-of-ide.(list-of-for-par.fpd-v))
  for-ref-par = list-of-ide.(list-of-for-par.fpd-r))
  for-par = for-val-par © for-ref-par
are-repetitions.for-par           → ‘formal par repetitions’
length.for-val-par ≠ length.apd-v → ‘incompatible numbers of value parameters’
length.for-ref-par ≠ length.apd-r → ‘incompatible numbers of reference parameters’
true                             → ‘OK’

```

In words, the lists of formal and actual parameter denotations of a procedure call are statically compatible if:

1. no formal parameter appears twice on a combined list of value- and reference parameters; a similar property of actual value-parameters is, of course, not required,
2. the mutually corresponding lists of formal and actual parameter denotations are of the same lengths.

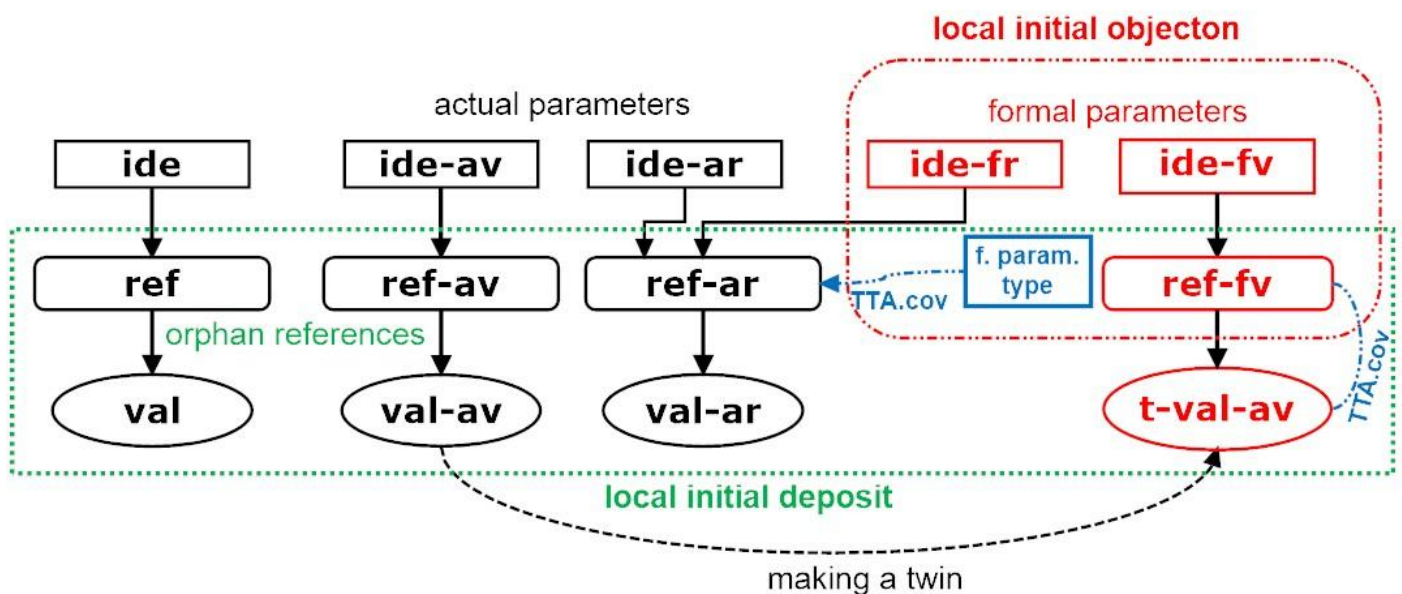
Note that empty lists of corresponding parameters are compatible.

The defined property is called *static* since it can be checked at compilation time, i.e., before program execution. Note that “static” does not mean “syntactic” — static compatibility requirement can’t be built into a grammar, and therefore it can’t be checked by a syntax analyzer.

#### 6.6.3.4 Passing actual parameters to a procedure

Function `pass-actual` describes the process of passing the values and references of actual parameters of a procedure call to the body of the called procedure. Technically it creates a local initial store to be elaborated by the body (Sec.6.6.3.2). Actual value-parameters must be declared and initialized, and actual reference-parameters must be declared, but not necessarily initialized.

Fig. 6.6-2 illustrates the mechanism of passing parameters. As was already shown in Sec. 6.6.3.2, the environment of the local input state will be a declaration-time environment whereas its store will be a local initial store created by our function.



**Fig. 6.6-2** Passing actual parameters to a procedure (a simplified picture)

We assume the following rules about the creation of local initial stores. All of them have an engineering character:

1. The only identifiers bound initially in the local store are formal parameters. Of course, during the execution of the call some local variables may be added (declared). As a consequence, procedures can't use global variables and, therefore, their only "side effects" are due to reference parameters.
2. The current references of actual reference-parameters **ref-ar** become the references of formal reference parameters **ide-fr**. To "meet the expectations" of procedure's designer, the actual types of **ide-ar**'s, i.e., the types of **ref-ar**'s, must be accepted by the declared types of formal parameters **ide-fr**'s.
3. Fresh references **ref-fv**'s are created for formal value-parameters. The types of these references are the declared types of formal parameters. To these references we assign the twins (see later) **t-val-av**'s of the values **val-av**'s of actual parameters. The new references must accept these twins. The picture shown in Fig. 6.6-2 is "simplified" since making a twin of an object requires replacing all tokens of its references by fresh ones, which may result in substantial modification (enrichment) of the deposit.
4. The origin tags of actual parameters become the origin tags of formal references, which seems to be an obvious choice. If an actual parameter is public, the corresponding formal parameter should be public as well. In turn, if it is private, then it should remain private with the unchanged origin in the local state.

To define a function of passing actual to formal parameters, we shall need a function that given a value returns its *twin*. A twin of a typed data is just this data, and a twin of an object is created by replacing all tokens in this object by fresh ones. Before we proceed to a formal definition of this function let's observe the following facts:

1. A description of an object requires the context of a deposit, and therefore our function must modify deposits.
2. The replacement of tokens by fresh tokens consumes tokens, and therefore our function must modify sets of free tokens.
3. Due to the fact that objects may include cycles, we have to stop the replacement of "old" tokens by new ones whenever we encounter a token that "is already new". To do this our function will "monitor" a set of new tokens.

To define the twining function we shall need two new domains:

|   |                                  |
|---|----------------------------------|
| <b>snt</b> : SetNewTok = Set.Token            | sets of new tokens <sup>43</sup> |
| <b>tot</b> : TupleOfTok = Token <sup>c*</sup> | tuples of tokens                 |

and two new functions:

|  |  |
|--|--|
| <b>tokens-of</b> : Value x Deposit                   | $\mapsto$ Set.Token                          |
| <b>get-new-tok</b> : Integer x SetNewTok x SetFreTok | $\mapsto$ TupleOfTok x SetNewTok x SetFreTok |

The first function given a value and a deposit returns the set of all tokens included in this value. We skip its obvious definition. The second generates a tuple of fresh tokens and appropriately modifies the sets of new tokens and free tokens. Its definition is the following:

```

get-new-tok.(n, snt, sft) =
  n ≤ 0    → 'number of tokens must be positive'
  n = 1    →
    let
      (tok, sft-1) = get-tok.sft
      snt-1       = snt | {tok}
      ((tok), snt-1, sft-1)
  n > 1    →
    let
      (tok, sft-1) = get-tok.sft
      snt-1       = snt | {tok}
      (tot, snt-2, sft-2) = get-new-tok.(n-1, snt-1, sft-1)

```

<sup>43</sup> Formally speaking we do not introduce here a new domain since SetNewTok = Set.Token = SetFreTok, but a new metaname to distinguish between two arguments of our function **snt** and **sft** that are of the same sort.

((tok) © tot, snt-2, sft-2)

The function that creates twins of values is the following:

create-twin : Value x Deposit x SetNewTok x SetFreTok  $\mapsto$   
 $\mapsto$  (Value x Deposit x SetNewTok x SetFreTok)

create-twin. (val, dep, snt, sft) =  
 val : TypDat  $\rightarrow$  (val, dep, snt, sft)  
 tokens-of.val  $\subseteq$  snt  $\rightarrow$  (val, dep, snt, sft)  
**let**  
 (obn, cl-ide) = val the argument value is an object  
 [ide-1/ref-1,...,ide-n/ref-n] = obn  
 (tok-i, prf-i) = ref-i  
 ((new-tok-1,...,new-tok-n), sft-1, snt-1) = get-new-tok.(n, snt, sft)  
 new-ref-i =  
   tok-i : snt  $\rightarrow$  ref-i  
   **true**  $\rightarrow$  (new-tok-i, prf-i) for i = 1;n  
 obn-1 = [ide-1/new-ref-1,...,ide-n/new-ref-n]  
 val-i = dep.ref-i for i = 1;n  
 (twin-val-1, dep-1, snt-2, sft-2) = create-twin.(val-1, dep, snt-1, sft-1)  
 ...  
 (twin-val-n, dep-n, snt-(n+1), sft-(n+1)) = create-twin.(val-n, dep-(n-1), snt-n, sft-n)  
 new-dep = dep-n[new-ref-1/twin-val-1,...,new-ref-n/twin-val-n]  
 twin-val = (obn-1, cl-ide)  
**true**  $\rightarrow$  (twin-val, dep-n, , snt-(n+1), sft-(n+1))

In the first step our function checks if the value *val* to be “twinned” is a typed data, and if this is the case, it returns the same value. Deposit and both sets of tokens remain unchanged and the process stops.

If the value is an object (obn, cl-ide), then we check if all tokens in this value have been already replaced, and if this is the case then the process stops.

Otherwise our function replaces all not-new references in *obn* by fresh references, and appropriately modifies the sets of new tokens and free tokens. Of course, in the first step of our recursion none of these references are new, but in further steps such a situation may happen, if there are cycles in our object.

After the first step of our recursion we have a new object *obn-1*, but all its references are dangling. Then, for every *val-i* assigned to a reference of the original *obn* we create recursively a twin of this value, and we assign this twin to the corresponding reference of *obn-1*. In each such step we appropriately modify the deposit and both sets of tokens “inherited” from the former step.

Nie wiem dlaczego, ale nie do końca jestem pewien definicji *create-twin*. Z drugiej jednak strony wydaje się dość oczywiste, że taka funkcja musi dać się dobrze zdefiniować. ???

Now we are ready to define the function of passing actual to formal parameters. Let *cl-ide* be the name of a class where our procedure is being declared:

pass-actual : ForParDen x ForParDen x ActParDen x ActParDen x Identifier  $\mapsto$   
 $\mapsto$  Env  $\mapsto$  Store  $\mapsto$  Store

pass-actual.(fpd-v, fpd-r, apd-v, apd-r, *cl-ide*).dt-env.ct-sto =  
 is-error.ct-sto  $\rightarrow$  ct-sto call time store  
 fpd-r = ()  $\rightarrow$  ‘list of formal reference parameters can’t be empty’

1. *checking the static compatibility of parameters*  
**let**  
   message = statically-compatible.(fpd-v, fpd-r, apd-v, apd-r)  
   message  $\neq$  ‘OK’  $\rightarrow$  ct-sto  $\blacktriangleleft$  message
2. *identifying the identifiers, values, and references of actual and formal parameters*  
**let**



for  $k, n \geq 0$   
 $((ide-fv.i, ted-fv.i) \mid i=1;k)$  = list-of-for-par.fpd-v (see Sec. 6.6.3.3)  
 $((ide-fr.i, ted-fr.i) \mid i=1;n)$  = list-of-for-par.fpd-r  
 $(ide-av.i \mid i=1;k)$  = apd-v  
 $(ide-ar.i \mid i=1;n)$  = apd-r  
 $(dt-cle, dt-pre, dt-cov)$  = dt-env declaration-time environment  
 $(ct-obn, ct-dep, ct-ota, ct-sft, 'OK')$  = ct-sto call-time store  
 $ct-obn.ide-av.i = ? \rightarrow ct-sto \blacktriangleleft$  'actual val. parameter not declared' for  $i = 1;k$   
 $ct-obn.ide-ar.i = ? \rightarrow ct-sto \blacktriangleleft$  'actual ref. parameter not declared' for  $i = 1;n$   
**let**  
 $ref-av.i = ct-obn.ide-av.i$  for  $i=1;k$  the references of actual value-parameters  
 $ref-ar.i = ct-obn.ide-ar.i$  for  $i=1;n$  the references of actual reference-parameters  
 $ct-dep.ref-av.i = ? \rightarrow ct-sto \blacktriangleleft$  'actual val. parameter not initialized'<sup>44</sup> for  $i = 1;k$   
**let**  
 $val-av.i$  =  $ct-dep.ref-av.i$  for  $i = 1;k$   
 $(dat-av.i, typ-av.i)$  =  $val-av.i$  for  $i = 1;k$   
 $(tok-av.i, (typ-rav.i, ota-av.i))$  =  $ref-av.i$  for  $i = 1;k$   
 $(tok-ar.i, (typ-rar.i, ota-ar.i))$  =  $ref-ar.i$  for  $i = 1;n$   
 3. *computing the types of formal parameters*  
**let**  
 $de-typ-fv.i = ted-fv.i(dt-env, ct-sto)$  declared types of formal value-parameters for  $i = 1;k$   
 $de-typ-fr.i = ted-fr.i(dt-env, ct-sto)$  declared types of formal reference-parameters for  $i = 1;n$   
 $de-typ-fv.i : Error \rightarrow ct-sto \blacktriangleleft d-typ-fv.i$  for  $i = 1;k$   
 $de-typ-fr.i : Error \rightarrow ct-sto \blacktriangleleft de-typ-fr.i$  for  $i = 1;n$   
 4. *creating twins of formal value-parameters*  
**let**  
 $sft-0$  =  $ct-sft$   
 $snt-0$  =  $\{ \}$   
 $dep-0$  =  $ct-dep$   
 $(twin-val-av.1, dep-1, snt-1, sft-1) = create-twin.(val-av.i, dep-0, snt-0, sft-0)$   
 $\dots$   
 $(twin-val-av.k, dep-k, snt-k, sft-k) = create-twin.(val-av.i, dep-(k-1), snt-(k-1), sft-(k-1))$   
 5. *creating references for formal value-parameters*  
**let**  
 $((tok-fv.1, \dots, tok-fv.k), snt, new-sft) = get-new-tok.(k, \{ \}, sft-k)$ <sup>45</sup>  
 $new-ref-fv.i$  =  $(tok-fv.i, (de-typ-fv.i, ota-av.i))$  for  $i = 1;k$   
 6. *checking type acceptance of:*  
*actual value-parameters by formal value-parameters*  
**not**  $de-typ-fv.i$  **TTA.dt-cov**  $typ-av.i \rightarrow sta \blacktriangleleft$  'value parameters not compatible' for  $i = 1;k$   
*actual reference-parameters by formal reference-parameters*  
**not**  $de-typ-fr.i$  **TTA.dt-cov**  $typ-rar.i \rightarrow sta \blacktriangleleft$  'reference parameters not compatible' for  $i = 1;n$   
 7. *creating a local initial objecton of the store*  
**let**  
 $li-obn-fv$  =  $[ide-fv.1/new-ref-fv.1, \dots, ide-fv.k/new-ref-fv.k]$  the binding of formal val-param.  
 $li-obn-fr$  =  $[ide-fr.1/ref-ar.1, \dots, ide-fr.n/ref-ar.n]$  the binding of formal reference-parameters  
 $li-obn$  =  $li-obn-fv \blacklozenge li-obn-fr$  local initial objecton

<sup>44</sup> Mathematically we could have assumed that not initialized value parameters are allowed, but such parameters would have not too much of a practical sense, since they would act as local variables. If, therefore, a not initialized parameter is passed to a procedure call, we may have a justified supposition that it is a programmer's mistake, rather than an intention. Consequently we signalize an error. The situation with reference parameters is, of course, different.

<sup>45</sup> In this place we use the function **get-new-tok** to generate tokens for the future reference of formal value parameters. In this context we do not need to monitor a set of new tokens but our function must be given such a set as an argument. Since it can be an arbitrary set, we choose the empty set  $\{ \}$  to play this role.

8. *creating a local initial object deposit*

li-dep = dep-k[new-ref-fv.1/twin-val-av.1,...,new-ref-fv.k/twin-val-av.k] bind. twins to new ref.

9. *creating a local initial store*

li-sto = (li-obn, li-dep, **cl-ide**, new-sft, 'OK')

10. *creating a local initial state*

li-sta = (dt-env, li-sto)

**true** → li-sta

Function **pass-actual** performs the following steps:

1. checks, if the list of formal reference parameters is not empty; although denotationally an empty list of formal parameter makes sense, a procedure with such list has not effect (is equivalent to **skip**), and therefore if such a procedure appears in the program, then it is most probably a mistake of a programmer,
2. checks the static compatibility of actual with formal parameters,
3. identifies/computes:
  - a. the identifiers of actual and formal parameters,
  - b. the references of actual parameters; if they are not declared, then an error is signaled,
  - c. the values of actual-value parameters; if they are not initialized, then an error is signaled; note that actual reference parameters need not be initialized,
4. computes the types of formal parameters; these types are computed in a state carrying a declaration-time environment, but the types declared in this environment (in its classes) are the same as the types in the call-time environment<sup>46</sup>,
5. creates twin values for formal value-parameters; for not-object values twins are just these values, whereas the twins of object differ from the source objects only in internal tokens (see Sec. 5.4.3),
6. creates new references for formal value-parameters; they get fresh tokens and declared types of formal parameters,
7. checks the acceptance for declaration-time covering relation of:
  - a. types **typ-av.i** of the values of actual value-parameters by the declared types **de-typ-fv.i** of corresponding formal parameters,
  - b. types **typ-rar.i** of the references of actual reference-parameters by the declared types **de-typ-fr.i** of corresponding formal parameters,
8. creates a local initial objecton by assigning created references to formal value-parameters, and the (old) references of actual reference-parameters to formal reference-parameters; the actual values are accepted by the new references, because they differ from the “old” references by tokens only,
9. creates a local initial deposit as an extension of the call-time deposit by new references bound to the values of actual value-parameters; note that all “old” references of actual value parameters remain bounded in the new deposit, but now they are orphan references, which means that we can’t access them from the local state,
10. creates a local initial store with new objecton, new deposit, new set of free tokens, and **cl-ide** (the name of the hosting class) as its origin tag (rule 6 in Sec. 5.4.3),
11. creates a local initial state by putting together the declaration-time environment with the local initial store.

Two remarks are necessary about the required compatibility between the profiles of actual and formal parameters:

- In the case of value parameters, the references of formal parameters **ref-fv**’s are created by the passing parameters mechanism (PPM) in such a way that the declared types of parameters become the types of these references. Then PPM assigns to these references the values of actual parameters, and therefore the types of these values — which may be different from the declared types — must be acceptable by the latter.
- In the case of reference parameters the situation is different. Now, PPM assigns the references of actual parameters **ref-ar**’s to formal parameters **ide-fr**’s. Then we only request that the types of **ref-ar**’s are accepted by the declared types. However, since **TTA.cov** is, by definition, transitive, also now declared types will accept the types of actual values.

<sup>46</sup> It is the consequence of our assumption of Sec. 6.3 that all class declarations precede in programs all instructions.

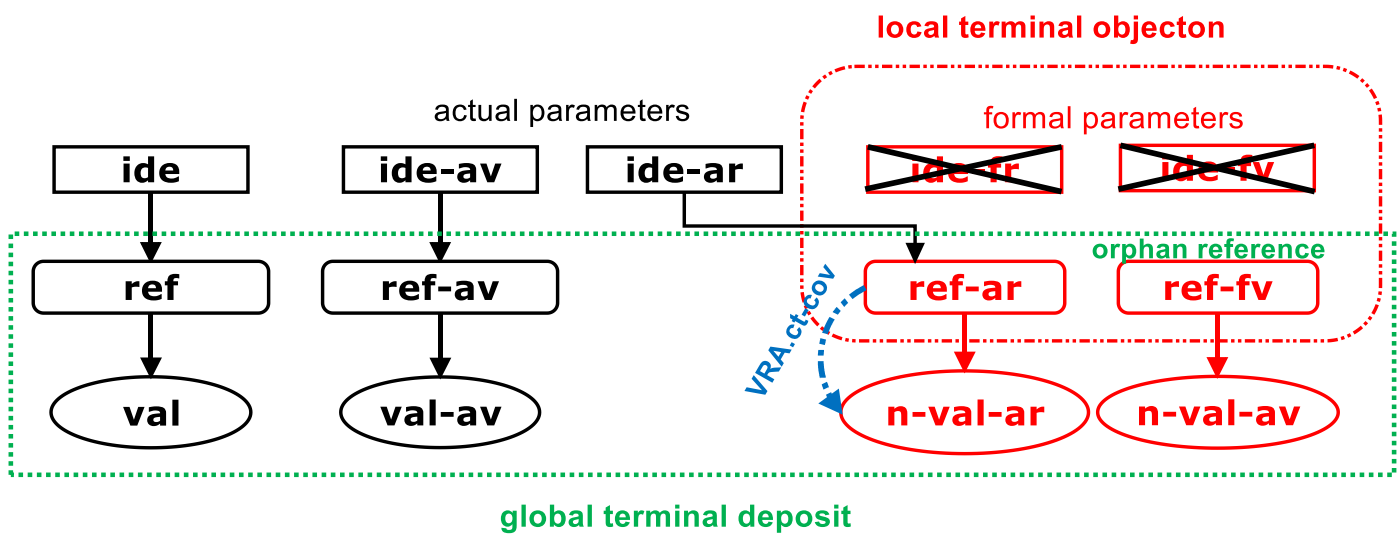


Notice that the described mechanism of creating local initial stores does not offer a possibility of using global variables/attributes, i.e. variables/attributes visible both outside and inside procedure-bodies. All “external interventions” of a procedure call must be realized by reference parameters, and therefore, must be explicitly declared. In our opinion such a solution contributes to the clarity of programs, and also simplifies construction rules of correct programs with procedure calls (cf. Sec. 9.4.6.3).

### 6.6.3.5 Returning the references of reference parameters

By the end of the execution of a procedure call we reach a *local terminal state* which consists of:

- a *local terminal store*, where the objecton binds formal parameters and possibly some local variables declared in the body of the procedure,
- a *local terminal environment*, where possibly new local classes, pre-procedures and procedures have been declared, and/or the covering relation has been augmented by new pairs of types.



**Fig. 6.6-3 Returning references to actual reference-parameters of a procedure**

Next, the exiting mechanism builds a *global terminal state* (Fig. 6.6-1) consisting of a call-time environment — all locally declared classes and procedures, and new pairs of types in covering relation cease to exist — and a *global terminal store* where actual reference parameters regain their call-time references. This store consists of (Fig. 6.6-3):

- *global call-time objecton* — where all global variables, including actual parameters, “regain visibility”, and all formal parameters and local variables cease to exist; actual reference-parameters point (back) to their call-time references,
- *local terminal deposit* — this deposit is passed unchanged to the global store, but the (created by the call) references of formal value-parameters and of local variables become orphan references; for the sake of simplicity we do not introduce a garbage-collection mechanism,
- *global call-time origin tag of the store* — note that the call-time origin tag may be: (1) either public-visibility origin tag \$, or (2) a class name; case (2) will happen, if our procedure was locally declared and called in the body of another procedure,
- *local terminal set of used references* — since we do not introduce a garbage collection mechanism, no formerly used tokens are released (simplification of the model),

Note in this place that the global terminal state which is of the form (ct-env, lt-sto) may (happen to) be not well-formed, if there is an actual reference-parameter **ide** such that in the local terminal state:

$$\text{ide} \rightarrow (\text{tok}, (\text{r-typ}, \text{ota})) \rightarrow (\text{dat}, \text{v-typ}) \quad \text{and} \quad (\text{r-typ}, \text{v-typ}) : \text{lt-cov}$$

but

$$(\text{r-typ}, \text{v-typ}) \text{ /: } \text{ct-cov}$$

This situation may happen since lt-cov may be larger than ct-cov. In such a case the mechanism of returning reference parameters must issue an error message. To detect such case we define an auxiliary function that for a given identifier and a given cov relation checks if the value of this identifier in a current state is **VRA.cov** acceptable in this state. Note that the argument cov may be different than the cov of the current state.

$\text{adequate} : \text{Identifier} \mapsto \text{CovRel} \mapsto \text{WfStore} \mapsto \{\text{tt}, \text{ff}\} \mid \text{Error}$

```
adequate.ide.cov.sto =
  is-error.sta      → error.sta
  let
    (obn, dep, ota, sft, 'OK') = sto
    obn.ide      = ? → 'parameter not declared'47
    dep.(obn.ide) = ? → 'parameter not initialized'
  let
    ref = obn.ide
    val = dep.ref
    ref VRA.cov val → tt
  true → ff
```

Having this function we can describe the mechanism of returning formal parameters:

$\text{return-formal} : \text{ForParDen} \mapsto \text{Store} \mapsto \text{Store} \mapsto \text{CovRel} \mapsto \text{Store}$

```
return-formal.fpd-r.ct-sto.lt-sto.ct-cov =
  is-error.lt-sto      → lt-sto
  let
    (ct-obn, ct-dep, ct-ota, ct-sft, 'OK') = ct-sto
    (lt-obn, lt-dep, lt-ota, lt-sft, 'OK') = lt-sto
    (ide-fr.i | i=1;n) = list-of-ide.(list-of-for-par.fpd-r)
    adequate.(ide-fr.i).ct-cov.lt-sto : Error → adequate.(ide-fr.i).ct-cov.lt-sto      for i = 1;n
    adequate.(ide-fr.i).ct-cov.lt-sto = ff → ct-sto ◀ 'ide-fr.i not adequate'          for i = 1;n
  true → (ct-obn, lt-dep, ct-ota, lt-sft, 'OK') for i = 1;n
```

The output store is a combination of:

- call-time objecton, and call-time origin tag,
- local-terminal deposit, and local-terminal set of free tokens.

Two facts are to be emphasized:

1. We do not allow a returned reference parameter to be not initialized condition A.). Note that this could have happened since we allow passing noninitialized actual reference parameters to formal parameters (cf. Sec. 6.6.3.4). However, whereas not initialized reference parameters at the entrance of a procedure call make sense, if the same happens at the exit, such parameters turn out to be useless. Since it is rational to expect that a programmer may introduce useless parameters only by mistake, we make an engineering decision to signalize an error whenever such a situation happens.

<sup>47</sup> In the context where we use function 'adequate' this case will never happen, still we have to take it into account to make our definition complete.

2. It could have happened that the value of a formal reference parameter `ide-fr.i` is of a type which was accepted by its reference for local covering relation, but is not accepted for global (call-time) relation. In such a case an error message should be generated. To do so, we use function `adequate`.

In point 2. we may see a problem, since formally this function gets a declaration-time environment (cf. Sec. 6.6.3.2), hence also a declaration-time covering relation `dt-cov`, and what we intend to do, is to check the adequacy for call-time covering relation `ct-cov`. Note, however, that by assumption made in Sec. 6.3, if a procedure is called prior to `open procedures`, then its call will abort, since our procedure is “not yet declared”, and therefore we do not need “to care” about covering relation. On the other hand, if procedure is called after `open procedures`, then we have the equality `ct-cov = dt-cov`, since no declaration may appear after open procedure.

### 6.6.3.6 Calling an imperative procedure

The calls of imperative procedures are atomic instructions. Their mechanism is described by the following constructor:

```

call-imp-pro : Identifier x Identifier x ActParDen x ActParDen  $\mapsto$  InsDen
call-imp-pro : Identifier x Identifier x ActParDen x ActParDen  $\mapsto$  WfState  $\rightarrow$  WfState
call-imp-pro.(cl-ide, pr-ide, apd-v, apd-r).ct-sta =
  is-error.ct-sta  $\rightarrow$  ct-sta
  let
    (ct-env, ct-sto) = ct-sta call-time state
    (ct-cle, ct-pre, ct-cov) = ct-env
    ct-pre.(cl-ide, pr-ide) = ?  $\rightarrow$  ct-sta  $\blacktriangleleft$  'procedure-unknown'
    ct-pre.(cl-ide, pr-ide)  $\neq$  ImpPro  $\rightarrow$  ct-sta  $\blacktriangleleft$  'imperative-procedure-expected'
  let
    ipr = ct-pre.(cl-ide, pr-ide)
    ipr.(apd-v, apd-r).ct-sto = ?  $\rightarrow$  ?
  let
    gt-sto = ipr.(apd-v, apd-r, ct-cov).ct-sto global terminal store
  true  $\rightarrow$  (ct-env, gt-sto)

```

To call a procedure, we first seek it in the procedure environment of a state (cf. Sec. 6.6.1), and then we apply it to the current (i.e. call-time) store. We recall that the terminal store may be the call-time store with an error (cf. Sec. 6.6.3.2).

## 6.6.4 Functional pre-procedures

### 6.6.4.1 The creation of a functional pre-procedure

Similarly to imperative pre-procedures, and for the same reason, we build *functional pre-procedures*. This process is formalized in the following definition:

```

create-fun-pre-pro : FunProSigDen x ProDen x ValExpDen x Identifier  $\mapsto$  FunPrePro
create-fun-pre-pro : FunProSigDen x ProDen x ValExpDen x Identifier  $\mapsto$ 
   $\mapsto$  Env  $\mapsto$  ActParDen  $\mapsto$  Store  $\rightarrow$  ValueE
create-fun-pre-pro.(fps, prd, ved, cl-ide).dt-env.apd.ct-sto = dt- creation time, ct- call time
  is-error.ct-sto  $\rightarrow$  error.ct-sto
  let
    (ct-obn, ct-dep, ct-sft, ct-ota, 'OK') = ct-sto
    (fpd, ted) = fps functional-procedure signature
    li-sto = pass-actual.(fpd, (), apd, (), cl-ide ).dt-env.ct-sto local initial store
  is-error.li-sto  $\rightarrow$  error.li-sto
  let
    li-sta = (dt-env, li-sto) local initial state
    ex-typ = ted.li-sta the expected type of the returned value

```

```

ex-typ : Error      → ex-typ
(prd • ved).li-sta = ?      → ?
(prd • ved).li-sta : Error → (prd • ved).li-sta
let
  (cor, typ) = (prd • ved).sta-li                               lt- local terminal
not ex-typ TTA.ct-cov typ → 'types-incompatible'
true                                → (cor, ex-typ)

```

This constructor is defined analogously to the corresponding constructor of imperative pre-procedures. The execution of its body consist in the evaluation of the argument value-expression **ved** in an output state of the argument program **prd**. In this way we get an intermediate value  $(cor, typ)$ , but the value finally returned by the procedure is  $(cor, ex\text{-}typ)$ , where  $ex\text{-}typ$  is the type expected by the procedure.

Note that we can't issue simply  $(cor, typ)$ , because, if  $typ$  has been locally declared, it will not be seen in the global environment. Of course, before we output  $(cor, ex\text{-}typ)$  we have to check if its type accepts  $typ$ .

#### 6.6.4.2 Calling a functional procedure

The constructor corresponding to the calls of deep functional procedures is the following:

```

call-fun-pro : Identifier x Identifier x ActParDen → ValExpDen
call-fun-pro : Identifier x Identifier x ActParDen → WfState → ValueE
call-fun-pro.(cl-ide, pr-ide, apd).ct-sta =                               ct- call time
  is-error.ct-sta      → ct-sta
let
  ((cle, pre, cov), ct-sto) = ct-sta
  pre.(cl-ide, pr-ide) = ?      → 'procedure-unknown'
  pre.(cl-ide, pr-ide) /: FunPro → 'functional-procedure-expected'
let
  fpr = pre.(cl-ide, pr-ide)
  fpr.apd.ct-sto = ?      → ?
true                                → fpr.apd.ct-sto

```

The called procedure is selected from the procedure environment, and then it is applied to the (call-time) store of the current state. If this application terminates successfully, then the outputted value becomes the output of the call. Note that  $fpr.apd.ct\text{-}sto$  may be an error.

### 6.6.5 Object pre-constructors

#### 6.6.5.1 Object constructors versus imperative procedures

Similarly as in many OO languages, objects are created in our model exclusively by dedicated imperative procedures called *object constructors*. For a class **MyClass** named '**MyClass**' an object constructor associated with this class (declared in this class) is a function that given a list of actual parameters, and the name **ide** of the future object, returns a store-to-store function that performs three major steps:

1. it creates an object of a class **MyClass** whose objecton is a twin of the objecton of **MyClass**, and whose type is '**MyClass**',
2. it (optionally) modifies the current deposit, by changing the values assigned to the attributes of the new object; to do this it uses a program,
3. it assigns the created object to the reference of **ide** in the deposit of the current store; the objecton of the new object is a sibling of the objecton of **MyClass**.

Since we do not want object constructors to have side effects (an engineering decision), we assume that they get only value parameters.

$oco : ObjCon = ActParDen \times Identifier \mapsto Store \rightarrow Store$

Since object constructors are regarded as procedures, by an analogy to pre-procedures, we introduce *object pre-constructors* with the following domain:

$$\text{opc} : \text{ObjPreCon} = \text{Env} \mapsto \text{ObjCon}$$

As we see, although the calls of object constructors are instructions, like the calls of imperative procedures, object constructors themselves are different from imperative procedures.

### 6.6.5.2 Creating an object pre-constructor

The creator of object pre-constructors given formal parameter denotations, a name of a class and a program denotation, returns an object pre-constructor. The latter given an environment returns an object constructor, that given the denotations of actual value parameters, an a name of the future object, and a (call time) store, returns a new store where the object name points to a new object of the type of the given class. The objecton of the created object is a sibling of the objecton of the involved class.

$$\text{create-obj-pre-con} : \text{ObjConSigDen} \times \text{ProDen} \mapsto \text{ObjPreCon}$$

$$\text{create-obj-pre-con} : \text{ForParDen} \times \text{Identifier} \times \text{ProDen} \mapsto$$

$$\mapsto \text{Env} \mapsto \text{ActParDen} \times \text{Identifier} \mapsto \text{Store} \rightarrow \text{Store}$$

$$\begin{aligned} \text{create-obj-pre-con}((\text{fpd}, \text{cl-ide}), \text{prd}).\text{dt-env}(\text{apd}, \text{ob-ide}).\text{ct-sto} = & \quad \text{cl-ide class identifier} \\ \text{is-error.ct-sto} \quad \quad \quad \rightarrow \text{ct-sto} & \quad \text{ob-ide object identifier} \end{aligned}$$

1. *parent class is identified*

**let**

$$(\text{dt-cle}, \text{dt-pre}, \text{dt-cov}) = \text{dt-env} \quad \text{declaration-time environment}$$

$$(\text{ct-obn}, \text{ct-dep}, \text{ct-ota}, \text{ct-sft}, \text{'OK'}) = \text{ct-sto} \quad \text{call-time store}$$

$$\text{dt-cle.cl-ide} = ? \quad \rightarrow \text{ct-sto} \blacktriangleleft \text{'parent class not declared'}$$

$$\text{ct-obn.ob-ide} = ? \quad \rightarrow \text{ct-sto} \blacktriangleleft \text{'object-identifier must be declared'}$$

$$\text{ct-dep}(\text{ct-obn.ob-ide}) = ! \quad \rightarrow \text{ct-sto} \blacktriangleleft \text{'object-identifier must not be initialized'}$$

2. *future reference of the constructed object is identified*

**let**

$$(\text{tok}, (\text{typ}, \text{ob-ota})) = \text{ct-obn.ob-ide} \quad \text{future reference of the constructed object}$$

$$(\text{ide}, \text{tye}, \text{mee}, \text{cl-obn}) = \text{dt-cle.cl-ide} \quad \text{parent class of the future object}$$

$$\text{not typ TTA.ct-cov cl-ide} \quad \rightarrow \text{'types not compatible'}$$

3. *formal-parameter store is created*

**let**

$$\text{fp-sto} = \text{pass-actual}(\text{fpd}, (), \text{apd}, (), \text{cl-ide}).\text{dt-env.ct-sto} \quad \text{formal-parameter store}$$

$$\text{is-error.fp-sto} \quad \rightarrow \text{ct-sto} \blacktriangleleft \text{error.fp-sto}$$

**let**

$$(\text{fp-obn}, \text{fp-dep}, \text{cl-ide}, \text{fp-sft}, \text{'OK'}) = \text{fp-sto}$$

$$\text{dom.cl-obn} \cap \text{dom.fp-obn} \neq \{\} \quad \rightarrow \text{ct-sto} \blacktriangleleft \text{'a clash between parameters and attributes'}$$

4. *local initial state is created*

**let**

$$(\text{tw-obn}, \text{li-sft}) = \text{create-twin}(\text{cl-obn}, \text{fp-sft}) \quad \text{tw-obn twin objecton}$$

$$\text{li-obn} = \text{fp-obn} \blacklozenge \text{tw-obn}$$

$$\text{li-sto} = (\text{li-obn}, \text{fp-dep}, \text{cl-ide}, \text{li-sft}, \text{'OK'})$$

$$\text{li-sta} = (\text{dt-env}, \text{li-sto}) \quad \text{local-initial state}$$

$$\text{prd.li-sta} = ? \quad \rightarrow ?$$

5. *local initial state is transformed into a local terminal state*

**let**

$$\text{lt-sta} = \text{prd.li-sta} \quad \text{local terminal state}$$

$$\text{is-error.lt-sta} \quad \rightarrow \text{ct-sto} \blacktriangleleft \text{error.lt-sta}$$

### 6. *resulting object and terminal global store are created*

```

let
  (lt-env, (lt-obn, lt-dep, lt-ota, lt-sft, 'OK')) = lt-sta
  re-obn  = lt-obn truncate-to dom.cl-obn           resulting objecton
  re-obj  = (re-obn, cl-ide)                        resulting object
  ob-ref  = ct-obn.ob-ide                           future reference of the resulting object
  tg-dep  = ct-dep[ob-ref/re-obj]                   terminal global deposit
true                                     → (ct-obn, tg-dep, ct-ota, lt-sft, 'OK')

```

Similarly to the former creators also this one builds a pre-constructor, that given a declaration-time environment returns a class constructor. The latter, given a call-time store returns this store with a resulting object bound in it. The resulting object is derived from the objecton of an indicated class, which is built into the pre-constructor. This action is performed in the following ten steps (we skip commenting errors):

1. A parent class indicated by its name **cl-ide** is identified.
2. The future reference of the new object is identified. It is the reference of the predeclared object variable **ob-ide**. The type of this reference must be compatible with the type of the created object, i.e. with **cl-ide**.
3. We create a formal-parameter store **fp-sto**, where only formal value parameters are bound in the objecton. The reference parameters are not involve, and the created store binds only formal value parameters.
4. The creation of a local initial state:
  - 4.1. We create a twin objecton **tw-obn** of the objecton **cl-obn** of the parent class.
  - 4.2. We create a local initial objecton **li-obn** by combining (overwriting) the twin objecton with the objecton of the formal-parameter store. The only attributes of this objecton are formal parameters and the attributes of the twin objecton. Note that these two sets of identifiers must be disjoint.
  - 4.3. We create a local initial store by combining the declaration-time environment with the local initial store. Note that the origin tag of this store is **cl-ide**.
  - 4.4. Local initial state **li-sta** is composed of the declaration-time environment, and the local initial store
5. The local initial state is transformed by the program **prd** to a local-terminal state **lt-sta**.
6. The creation of the resulting object and terminal global store
  - 6.1. The local terminal objecton is truncated to the attributes of the objecton of the class **cl-obn** thus giving a resulting objecton **re-obn**. This objecton is then used to create the resulting object **re-obj** by adding to it its type **cl-ide**.
  - 6.2. A terminal global deposit of the store is created by binding the created object to its reference **ob-ref**. Note that the compatibility of the types of **ob-ide** and **re-obj** has been checked in point 2.
  - 6.3. A terminal global deposit **tg-dep** is created by assigning the created object to **ob-ide** in the call-time deposit.
  - 6.4. The terminal global store includes the call-time objecton, the new deposit, the call-time covering relation (an engineering decision), a call-time origin tag, and a local terminal set of free tokens.

Returning to our example of an object constructor **ConstructObject** of Sec. 5.1, the program involved in the creation of a new object by this constructor consists of a pair of assignment statements:

```

no    := number + 1;
next  := node

```

where **no** and **next** are object's attributes, and **number** and **node** are formal parameters.

#### 6.6.5.3 Calling object constructors

Calls of object constructors are instructions that create new objects and assign them to predeclared variables. The constructor of such calls takes a name of a class **cl-ide** where the a constructor has been declared, the name **co-**



ide of this constructor, an identifier *ob-ide* to which the new object will be assigned, a list of actual value parameters, and returns an instruction denotation:

```

call-obj-con : Identifier x Identifier x Identifier x ActParDen  $\mapsto$  InsDen
call-obj-con : Identifier x Identifier x Identifier x ActParDen  $\mapsto$  WfState  $\rightarrow$  WfState
call-obj-con.(ob-ide, cl-ide, co-ide, apd-v).sta =
  is-error.ct-sta  $\rightarrow$  ct-ct-sta
  let
    ((ct-cle, ct-pre), ct-sto) = ct-sta call-time state
    ct-pre.(cl-ide, co-ide) = ?  $\rightarrow$  ct-sta  $\blacktriangleleft$  'constructor unknown'
    ct-pre.(cl-ide, co-ide)  $\neq$  ObjCon  $\rightarrow$  ct-sta  $\blacktriangleleft$  'object constructor expected'
  let
    oco = ct-pre.(cl-ide, co-ide)
    oco.(apd-v, ob-ide).ct-sto = ?  $\rightarrow$  ?
  let
    new-sto = oco.(apd-v, ob-ide).ct-sto
  true  $\rightarrow$  (ct-env, new-sto)

```

## 6.7 Declarations

### 6.7.1 An overview of declarations

Declarations in our model may act at two different levels:

1. at the **level of states**
  - a. they assign references, classes and procedures to identifiers,
  - b. they assign values to references in deposits.
  - c. they modify covering relations,
2. at the **level of classes**
  - a. they assign references, pre-procedures and types to identifiers,
  - b. they assign values to references in deposits.

The assignment of a reference to an identifier is usually referred to as a *variable* or an *attribute declaration*.

The fact that type declarations have been “located” in classes, rather than at the level of states, has an engineering character. It is pretty natural that classes should have an option of having their own types. Although we could have introduced “global” declarations of types at the level of states, we are not doing so since types, similar to procedures, will be public. Of course, if it ever comes to implementing **Lingua**, global declarations of types may be introduced, but from an “educational perspective” of our book, they would be redundant.

The declarations of classes will be executed in three steps:

- the choice of a parent class which may be either empty or previously declared,
- the removal from this class of all object pre-constructors (an engineering decision<sup>48</sup>)
- an enrichment of the resulting class by new items with the help of class transformers.

At the stage of class transformations we may:

- add abstract items,
- concretize abstract items, i.e. replace abstract items by corresponding concrete ones,
- add concrete items.

---

<sup>48</sup> It is usual in the existing programming languages that object constructors are not inherited by children classes from their parent classes. This rule seems rather obvious — when we create an object of a class, we use an object constructor of that class.





```

loi = ()                                → sta ◀ 'empty list of variables can't be declared'
let
  ((cle, pre, cov), (obn, dep, ota, sft, 'OK')) = sta
  (ide-1,...,ide-n)                             = loi
ted.sta : Error                               → sta ◀ ted.sta
declared.ide-i.sta                             → sta ◀ 'identifier declared'      for i = 1;n
yed.sta = ?                                    → ?
let
  de-typ      = ted.sta                                declared type
  (tok-1, sft-1) = get-tok.sft
  (tok-i, sft-i) = get-tok.sft(i-1)                    for i = 2;n
  ref-i        = (tok-i, (de-typ, $))
de-typ : ObjTyp and cle.de-typ = ? → 'class unknown'
true      → ((cle, pre, cov), (obn[ide-i/ref-i | i = 1;n], dep, ota, sft-n, 'OK'))

```

Note that if we declare an object variable, we check if the corresponding type is a declared name of a class. This rule is in contrast to the case where we add an abstract object-attribute to a class. In that case, as we are going to see in Sec. 6.7.4.2, we shall allow that the type of an attribute, i.e. the name of a corresponding class, may be not (yet) declared. Such a solution is necessary to allow anticipatory referencing in classes.

Another fact to be noted is that variables are always public (an engineering decision), which means that their origin tags are equal to \$.

### 6.7.3 Declarations of classes — a basic constructor

As has been already said, classes are declared in three steps:

1. In the first step we identify an initial *parent class* which is either an empty class or an earlier declared one.
2. In the second step we generate a *funding class*. If parent class was empty then the funding class is just this empty class. Otherwise we take a declared earlier parent class and remove from it all types, pre-procedures and object pre-constructors (an engineering decision). We also replace its name by a new one. What is inherited by funding class from parent class are, therefore, its attribute and signatures<sup>49</sup>. Note, however, that the inheritance of attributes means also the inheritance of their references, and therefore also values assigned to them in the deposit. On the other hand, as we are going to see later, the initial values of attributes may be changed at later stages of the declaration execution.
3. In the third step, we apply a class transformer (Sec. 6.7.4) to enrich funding class by new attributes, types, procedure signatures and pre-procedures.

We recall (Sec. 6.1) that the domain of class-transformer denotations is the following:

$\text{ctd} : \text{ClaTraDen} = \text{Identifier} \mapsto \text{WfState} \rightarrow \text{WfState}.$

Here some methodological remark are in order to explain why class transformers modify states rather than classes, and why they take an identifier as an argument?

In an earlier version of our model class transformers were functions transforming classes and states, rather than classes “in states”:

$\text{ClaTraDen} = (\text{Class} \times \text{WfState}) \rightarrow (\text{Class} \times \text{WfState})$

The modification of states by class transformers is necessary to describe the initialization of class attributes, whose values are created by the evaluation of value expressions. From a denotational perspective, this model worked quite well. Still, as turned out later, it led to technical problems in the definitions of rules for the creation

<sup>49</sup> In typical programming languages (??? Janusza prosimy o przykłady) funding class inherits from parent class all items except object pre-constructors. However, since in our model we have assumed (cf. Sec. 5.1) that types and procedures are public, we may access them independently of the class where they have been declared. Making their copies would not have much of a practical sense.

of correct class declarations (Sec. 9.4.4). This fact shows that in designing a programming language, we should consider not only the simplicity of its denotational model but also an ease of building program-construction rules (cf. Sec. 3.3).

Once we have assumed that class transformers will not get “input classes” as their arguments, we had to indicate these classes in a different way. Class identifiers were an obvious choice to play this role.

In the end, to define our constructor of class-declaration denotations we shall need an auxiliary function to create funding classes from parent classes:

```

make-funding-class : Clalnd  $\mapsto$  Identifier  $\mapsto$  WfState  $\mapsto$  Class | Error
make-funding-class.cli.ide.sta =
  cli = 'empty-class'  $\rightarrow$  (ide, [ ], [ ], [ ])
  let
    ((cle, pre, cov), sto) = sta
    cle.cli = ?  $\rightarrow$  'parent class unknown'
  let
    (cl-ide, tye, mee, obn) = cla.cli          parent class
    [ide-1/ppr-1,...,ide-n/ppr-n, ide-(n+1)/sig-1,...,ide-(n+k)/sig-k] = mee
  true  $\rightarrow$  (ide, [ ], [ide-(n+1)/sig-1,...,ide-(n+k)/sig-k], obn)

```

If the class indicator is ‘empty-class’, then the funding class is an empty class with *ide* as its internal name. Otherwise the funding class in the class indicated by *cli* and modified by giving it *ide* as a new name, removing all types from its type environment, and removing all pre-procedures from its method environment. The constructor of class-declaration denotations is now the following:

```

cla-dec : Identifier x Clalnd x ClaTraDen  $\mapsto$  DecDen
cla-dec : Identifier x Clalnd x ClaTraDen  $\mapsto$  WfState  $\mapsto$  WfState
cla-dec .(de-ide, pa-cli, ctd).sta =          de- for “declared”; pa- for “parent”
  is-error.sta  $\rightarrow$  sta
  declared.de-ide.sta  $\rightarrow$  sta  $\blacktriangleleft$  'identifier already declared'
  let
    ((cle, pre, cov), sto) = sta
    fu-cla = make-funding-class.pa-cli.de-ide.sta          funding class
  fu-cla : Error  $\rightarrow$  sta  $\blacktriangleleft$  fu-cla
  let
    fu-sta = ((cle[de-ide/fu-cla], pre, cov), sto)          funding state
  ctd.de-ide.fu-sta = ?  $\rightarrow$  ?
  let
    res-sta = ctd.de-ide.fu-sta          resulting state (with an enriched funding class)
  is-error.res-sta  $\rightarrow$  sta  $\blacktriangleleft$  error.res-sta
  true  $\rightarrow$  res-sta

```

The declaration of a class starts from making a funding class, and assigning it *de-ide* to class environment in an intermediate state called a *funding state*. This state, hence the funding class in this state, is then modified by the argument class transformer *ctd*. Note that the first argument of *ctd* is *de-cla*, i.e., the name of the modified class.

## 6.7.4 Class transformers

### 6.7.4.1 The signatures of constructors

Classes are transformed by adding to them new attributes, types or methods. Technically we bind new identifiers in class objectons, in type environments or in method environments. In all cases we have three options: we can add an abstract item, a concrete item, or we can concretize an abstract item. The last option may be used when we build a child of an earlier declared class. Finally, class transformers may be composed sequentially.

The domain of class-transformer denotations has been defined in Sec. 6.7.3. The list of constructors of class-transformer denotations, is the following, where **abs** means “abstract” and **con** means “concrete”.

|                           |  |                     |
|---------------------------|--|---------------------|
| <b>add-abs-att</b>        | : Identifier x TypExpDen x PriSta                | $\mapsto$ ClaTraDen |
| <b>concretize-abs-att</b> | : Identifier x ValExpDen                         | $\mapsto$ ClaTraDen |
| <b>add-con-att</b>        | : Identifier x ValExpDen x TypExpDen x PriSta    | $\mapsto$ ClaTraDen |
| <b>add-abs-typ</b>        | : Identifier                                     | $\mapsto$ ClaTraDen |
| <b>concretize-typ</b>     | : Identifier x TypExpDen                         | $\mapsto$ ClaTraDen |
| <b>add-con-typ</b>        | : Identifier x TypExpDen                         | $\mapsto$ ClaTraDen |
| <b>add-abs-imp-met</b>    | : Identifier x ImpProSigDen                      | $\mapsto$ ClaTraDen |
| <b>concretize-imp-met</b> | : Identifier x ImpProSigDen x ProDen             | $\mapsto$ ClaTraDen |
| <b>add-con-imp-met</b>    | : Identifier x ImpProSigDen x ProDen             | $\mapsto$ ClaTraDen |
| <b>add-abs-fun-met</b>    | : Identifier x FunProSigDen                      | $\mapsto$ ClaTraDen |
| <b>concretize-fun-met</b> | : Identifier x FunProSigDen x ProDen x ValExpDen | $\mapsto$ ClaTraDen |
| <b>add-con-fun-met</b>    | : Identifier x FunProSigDen x ProDen x ValExpDen | $\mapsto$ ClaTraDen |
| <b>add-abs-obj-met</b>    | : Identifier x ObjConSigDen                      | $\mapsto$ ClaTraDen |
| <b>concretize-obj-met</b> | : Identifier x ObjConSigDen x ProDen             | $\mapsto$ ClaTraDen |
| <b>add-con-obj-met</b>    | : Identifier x ObjConSigDen x ProDen             | $\mapsto$ ClaTraDen |
| <b>compose-cla-tra</b>    | : ClaTraDen x ClaTraDen                          | $\mapsto$ ClaTraDen |

In the following sections, we will show some examples of the definitions of these constructors. Each of these constructors will perform three similar steps:

1. it will identify a class assigned to **cl-ide**,
2. it will appropriately modify this class,
3. it will assign the new class to **cl-ide** in the class environment of the current state.

Transformers built by the first fifteen constructors will be called *atomic transformers*, to contrast them from *composed transformers* built by means of **compose-cla-tra**.

#### 6.7.4.2 Adding an abstract attribute to the objecton of a class

Contrary to variables that are, as a rule, private (Sec. 6.7.2), when we declare a class attribute we have to decide about its visibility status. To incorporate the privacy mechanism into the declarations of class attributes, we introduce a new that includes two marks defining *privacy status*

**pst** : PriSta = {'private', 'public'}.

Adding an abstract attribute to a class consists in adding a new attribute to class objecton. The corresponding constructor is similar to the declaration of a variable, except that now we decide about the visibility status of the new attribute. The resulting class transformer enriches a class named **cl-ide** by a new attribute **at-ide**.

**add-abs-att** : Identifier x TypExpDen x PriSta  $\mapsto$  ClaTraDen     i.e.  
**add-abs-att** : Identifier x TypExpDen x PriSta  $\mapsto$  Identifier  $\mapsto$  WfState  $\rightarrow$  WfState  
**add-abs-att**.(**at-ide**, **ted**, **pst**).**cl-ide**.sta =  
   **is-error**.sta      $\rightarrow$  sta  
   **let**  
     ((**cle**, **pre**, **cov**), (**obn**, **dep**, **ota**, **sft**, 'OK')) = sta  
   **cle**.**cl-ide** = ?      $\rightarrow$  'class unknown'  
   **declared**.**at-ide**.sta      $\rightarrow$  sta  $\blacktriangleleft$  'attribute already declared'  
   **ted**.sta : Error      $\rightarrow$  sta  $\blacktriangleleft$  **ted**.sta  
   **let**  
     (**cl-ide**, **tye**, **mee**, **cl-obn**) = **cle**.**cl-ide**  
     **de-typ** = **ted**.sta  
     (**tok**, **new-sft**) = **get-tok**.sft  
     **ref** =

de-typ – declared type

```

pst = 'public' → (tok, (de-typ, $))
pst = 'private' → (tok, (de-typ, cl-ide))
new-cla = (cl-ide, tye, mee, cl-obn[at-ide/ref])
true → ((cle[cl-ide/new-cla], pre, cov), (obn, dep, ota, new-sft, 'OK'))

```

Observe that in this definition we use the fact that the input state is well-formed, and therefore the internal name of a class, here `cl-ide`, is a component of this class.

Two more facts are to be noted that make abstract attribute declarations different from the declarations of variables (Sec. 6.7.2):

First, an attribute may be private, in which case its origin tag is equal to the name of the hosting class, rather than to `$`.

Second, if the declared type `de-typ` is an object type, in which case it is supposed to be a name of a class, we do not check if this is really the case, thus allowing to define abstract object-attributes with an *anticipatory referencing* to a class that hasn't been declared yet. This situation is illustrated by two examples written in Java (Fig. 6.7-1), where — additionally — we have to do with a class recursion.

|   |   |
|---|---|
| <pre> <b>class</b> ListNode{   int      no = 2;   ListNode next;   ... } </pre> | <pre> <b>class</b> A{   B b;   void b(B b){     this.b = b; } } <b>class</b> B{   A a = new A(); ... } </pre> |
| Case A: simple recursion  | Case B: mutual recursion  |

**Fig. 6.7-1** Two examples of recursive anticipatory referencing in Java

In **Case A** class `ListNode` refers recursively to itself, in **Case B** class `A` refers to `B`, and `B` refers to `A`. Note that if in the second case class `B` would not refer to class `A`, then there would be no recursion but still an anticipatory referencing will be the case.

Since classes are regarded as types of objects, a question arises if we should allow anticipatory referencing between datatypes as well.

#### 6.7.4.3 Adding a concrete attribute to the objecton of a class

`add-con-att` : Identifier x ValExpDen x TypExpDen x PriSta  $\mapsto$  ClaTraDen i.e.

`add-con-att` : Identifier x ValExpDen x TypExpDen x PriSta  $\mapsto$  Identifier  $\mapsto$  WfState  $\rightarrow$  WfState

`add-con-att.(at-ide, ved, ted, pst).cl-ide.sta =`

```

is-error.sta      → sta
declared.at-ide.sta → sta ◀ 'attribute already declared'
ted.sta : Error    → sta ◀ ted.sta
ved.sta = ?        → ?
ved.sta : Error    → sta ◀ ved.sta

```

**let**

((cle, pre, cov), (obn, dep, ota, sft, 'OK')) = sta

cle.cl-ide → 'class unknown'

**let**

```

(cle, pre, cov) = cle.cl-ide
de-typ          = ted.sta
de-val          = ved.sta
(tok, new-sft)  = get-tok.sft
ref             =
pst = 'public'  → (tok, (de-typ, $))

```

de-typ – declared type

```

pst = 'private' → (tok, (de-typ, cl-ide))
new-cla = (cl-ide, tye, mee, cl-obn[at-ide/ref])
true → ((cle[cl-ide/new-cla], pre, cov), (obn, dep[ref/de-val], ota, new-sft, 'OK'))

```

#### 6.7.4.4 Concretizing abstract attributes and adding concrete attributes

Concretizations of an abstract attributes acts as assignments (Sec. 6.5.2) except that the concretized attribute must be abstract. In other words, we do not allow for a replacement of a value of an attribute by an new one at the stage of a class declaration (an engineering decision). We skip formal definitions of this constructors.

#### 6.7.4.5 Adding a type constant to a class

The case of adding a type constant to a type environment includes three constructors (cf. Sec. 6.7.4.1): adding an abstract type, concretizing an abstract type and adding a concrete type. In the first case we add a type constant with a pseudotype  $\Theta$  assigned to it.

```

add-abs-typ : Identifier → ClaTraDen
add-abs-typ : Identifier → Identifier → WfState → WfState
add-abs-typ.ty-ide.cl-ide.sta =
  is-error.sta → sta
  declared.ty-ide.sta → sta ◀ 'type name declared in state'
  let
    ((cle, pre, cov), sto) = sta
  cle.cl-ide = ? → sta ◀ 'class unknown'
  let
    (cl-ide, tye, mee, obn) = cle.cl-cla
    new-cla = (cl-ide, tye[ty-ide/Θ], mee, obn)
  true → ((cle[cl-ide/new-cla], pre, cov), sto)

```

In the second case we concretize an abstract type:

```

concretize-typ : Identifier x TypExpDen → ClaTraDen
concretize-typ : Identifier x TypExpDen → Identifier → WfState → WfState
concretize-typ.(ty-ide, ted).cl-ide.sta =
  is-error.sta → sta
  let
    ((cle, pre, cov), sto) = sta
    cle.cl-ide = ? → sta ◀ 'class unknown'
  let
    (cl-ide, tye, mee, obn) = cle.cl-ide
    tye.ty-ide = ? → sta ◀ 'type name unknown'
    tye.ty-ide ≠ Θ → sta ◀ 'only abstract types may be concretized'
    ted.sta : Error → sta ◀ ted.sta
  let
    typ = ted.sta
    new-cla = (cl-ide, tye[ty-ide/typ], mee, obn)
    typ /: ObjTyp → ((cle[cl-ide/new-cla], pre, cov), sto)
    cle.typ = ? → sta ◀ 'object type unknown'
  true → ((cle[cl-ide/new-cla], pre, cov), sto)

```

The following two conditions must be satisfied to concretize a type constant `ty-ide`:

1. `ty-ide` must be declared as an abstract type constant; i.e., we can't change a type assigned to a concrete type constant, nor we can concretize a not declared constant,
2. if the type to be assigned to `ty-ide` is an object type, it must be the name of a declared class

The case of adding a concrete type is analogous — it combines adding a new abstract type with concretizing it to a given type.



### 6.7.4.6 Adding a method constant to a class

In a method environment of a class we can bind three categories of pre-procedures — imperative and functional pre-procedures, and object pre-constructors — and three corresponding categories of signatures. We can also concretize abstract methods by completing signatures to pre-procedures. Since all the corresponding definitions are quite simple and analogous to each other, we show only three examples of such constructors. We start from introducing an auxiliary function:

**get-parameters** : ForParDen  $\mapsto$  Sub.Identifier

which given a (list of) formal parameters returns the set of identifiers included in these parameters. We skip a formal definition of this function.

Our first constructor builds the denotation of a declaration of an imperative pre-procedure:

**add-con-imp-met** : Identifier x ImpProSigDen x ProDen  $\mapsto$  ClaTraDen i.e.  
**add-con-imp-met** : Identifier x ForParDen x ForParDen x ProDen  $\mapsto$   
 $\mapsto$  Identifier  $\rightarrow$  WfState  $\rightarrow$  WfState  
**add-con-imp-met**.(pr-ide, fpd-v, fpd-r, prd).cl-ide.sta =  
 is-error.sta  $\rightarrow$  sta  
**let**  
 ((cle, pre, cov), sto) = sta  
 (v-ide-1, ..., v-ide-n) = get-parameters.fpd-v  
 (r-ide-1, ..., r-ide-k) = get-parameters.fpd-r  
 for-par = (v-ide-1, ..., v-ide-n)  $\odot$  (r-ide-1, ..., r-ide-k)  
 cle.cl-ide = ?  $\rightarrow$  sta  $\triangleleft$  'class unknown'  
 are-repetitions.for-par  $\rightarrow$  sta  $\triangleleft$  'formal par repetitions'  
 declared.pr-ide.sta  $\rightarrow$  sta  $\triangleleft$  'identifier not free'  
 declared.v-ide-i.sta  $\rightarrow$  sta  $\triangleleft$  'identifier not free' for i = 1;n<sup>50</sup>  
 declared.r-ide-i.sta  $\rightarrow$  sta  $\triangleleft$  'identifier not free' for i = 1;k  
**let**  
 (cl-ide, tye, mee, obn) = cle.cl-ide  
**let**  
 ipp = create-imp-pre-pro.(fpd-v, fpd-r, prd, cl-ide)  
 new-cla = (ide, tye, mee[pr-ide/ipp], obn)  
**true**  $\rightarrow$  ((cle[cl-ide/new-cla], pre, cov), sto)

The second constructor corresponds to concretizing a previously declared functional method:

**concretize-fun-met** : Identifier x FunProSigDen x ProDen x ValExpDen  $\mapsto$  ClaTraDen  
**concretize-fun-met** : Identifier x FunProSigDen x ProDen x ValExpDen  $\mapsto$   
 $\mapsto$  Identifier  $\rightarrow$  WfState  $\rightarrow$  WfState  
**concretize-fun-met**.(pr-ide, fps, prd, ved).cl-ide.sta =  
 is-error.sta  $\rightarrow$  sta  
**let**  
 ((cle, pre, cov), sto) = sta  
 cle.cl-ide = ?  $\rightarrow$  sta  $\triangleleft$  'class unknown'  
**let**  
 (cl-ide, tye, mee, obn) = cle.cl-ide  
 mee.pr-ide = ?  $\rightarrow$  sta  $\triangleleft$  'method unknown'  
 mee.pr-ide /: FunProSigDen  $\rightarrow$  sta  $\triangleleft$  'signature of functional procedure expected'  
 fps  $\neq$  mee.pr-ide  $\rightarrow$  sta  $\triangleleft$  'signatures not compatible'  
**let**  
 fpp = create-fun-pre-pro.(fps, prd, ved, cl-ide)

<sup>50</sup> Denotationally it is not necessary that formal parameters are free, but we take this assumption since it technically simplifies future rules of correct program development (cf. Sec. 9.4.6.3).



```

new-cla    = (cl-ide, tye, mee[pr-ide/fpp], obn)
true       → ((cle[cl-ide/new-cla], pre, cov), sto)

```

Third constructor corresponds to a declaration of a concrete object constructor.

```

add-con-obj-met : Identifier x ObjConSigDen x ProDen → ClaTraDen
add-con-obj-met : Identifier x ForParDen x Identifier x ProDen →
                                     Identifier → WfState → WfState
add-con-obj-met.(oc-ide, fpd, cl-ide, prd).cl-ide.sta =          oc- object-constructor
is-error.sta      → sta
let
  ((cle, pre, cov), sto) = sta
  (ide-1, ..., ide-n)    = get-parameters.fpd
  declared.oc-ide.sta    → 'identifier not free'
  cle.cl-ide = ?         → sta ◀ 'class unknown'
  declared.ide-i.sta     → sta ◀ 'identifier not free'    for i = 1;n
let
  (cl-ide, tye, mee, obn) = cle.cl-ide
  opc                    = create-obj-pre-con.((fpd, cl-ide), prd)
  new-cla                = (cl-ide, tye, mee[oc-ide/opc], obn)
true                     → ((cle[cl-ide/new-cla], pre, cov), sto)

```

#### 6.7.4.7 Composing transformers sequentially

The following constructor simply combines the “declaration layers” of transformers sequentially:

```

compose-cla-tra : ClaTraDen x ClaTraDen → ClaTraDen
compose-cla-tra.(cdt-1, cdt-2).ide = (ctr-1.ide) • (ctr-2.ide)

```

Intuitively speaking a sequential composition of transformers describes a process of a cumulative creation of a class named *ide*, provided that it is declared in the argument state. In this process a class assigned to *ide* (if any) is modified to a new class by adding to it some new items. Note that this process is possible only “internally” within a class declaration, since this is the only context in which we can use class transformers. It can’t be performed “externally” since class transformers do not belong to the category of declarations. Consequently, a class once declared, can’t be changed in the future.

Note also that *compose-cla-tra* is associative, since *•* is associative.

### 6.7.5 Enrichments of covering relations

Our last category of declarations are enrichments of covering relations. Although they refer to types, that are “stored” in classes, the enrichments of covering relations do not transform classes, but environments. This is an engineering decision which makes covering relations globally accessible.

```

enrich-cov-rel : TypExpDen x TypExpDen → DecDen          i.e.
enrich-cov-rel : TypExpDen x TypExpDen → WfState → WfState
enrich-cov-rel.(ted-1, ted-2).sta =
  is-error.sta      → sta
  ted-i.sta : Error  → sta ◀ ted-i                        for i = 1,2
let
  typ-i                    = ted-i.sta                    for i = 1,2
  ((cle, pre, cov), (obn, dep, ota, sft, 'OK')) = sta
  cov-1                    = enrich-cov.(cov, typ-1, typ-2)
cov-1 : Error          → sta ◀ cov-1
typ-1, typ-2 : ObjTyp → ((cle, pre, cov-1), (obn, dep, ota, sft, 'OK'))
true                   →
let

```

|   |             |
|---|-------------|
| ide-i = typ-i   | for i = 1,2 |
| cle.ide-i = ? → 'object types must point to declared classes' | for i = 1,2 |
| true → ((cle, pre, cov-1), (obn, dep, ota, sft, 'OK'))        |             |

We recall that `enrich-cov` (Sec. 5.4.2) realizes the requirement that if one of the types is an object type, then so must be the other. Additionally our constructor checks if object types point to declared classes.

A word of comment is necessary here to explain why the enrichments of covering relations were included in the category of declarations rather than instructions. The reason is to ensure that once program execution passes the declaration part of the program (cf. Sec. 6.3), the covering relation won't be changed. This assumption simplifies the future rule of creating correct procedure calls (see Sec. 9.4.6.3).

## 6.7.6 The openings of procedures

As we have assumed in Sec. 6.3 every program in our language is a sequential combination of three components:

1. a declaration, possibly composed,
2. a single predefined *procedures' opening*,
3. an instruction, also possibly composed

The assumption 2. means that the domain of procedures' openings includes only one element

`pod : ProOpeDen = {open-pro-den}`

where

`open-pro-den : WfState → WfState,`

and that this element is built by a zero-argument constructor

`create-open-pro-den : → ProOpeDen`

i.e. it is built by language designer, rather than by programmers, as that is the case with declarations and instructions. To incorporate opening declarations into our model, we first introduce an auxiliary function

`get-pre-pro : WfState → (ProInd x PrePro)c*` get pre-procedures

This function given a state  $((cle, pre, cov), sto)$ , returns a sequence of all pairs  $((cl-ide, pr-ide), prp)$  where:

- `pr-ide` is a name of a pre-procedure declared in a class named `cl-ide`,
- `prp` is the corresponding pre-procedure.

We skip a formal definition of this function, and we recall (cf. Sec. 5.3) that the pairs of identifiers  $(cl-ide, pr-ide)$  are called procedure indicators. Now, a half-formal definition of our constructor is the following:

```

create-open-pro-den : → ProOpeDen
create-open-pro-den : → WfState → WfState
create-open-pro-den .().dt-sta = dt-sta declaration-time state
  is-error.dt-sta           → dt-sta
  get-pre-pro.dt-sta = ()   → dt-sta ◀ 'no procedures to declare'
  let
    ((pri-1, prp-1), ..., (pri-n, prp-n)) = get-pre-pro.dt-sta
    ((dt-cle, dt-pre, dt-cov), dt-sto)    = dt-sta
    pro-1                                = prp-1.(dt-cle, dt-pre[pri-1/pro-1, ..., (pri-n/pro-n), dt-cov]
    ...
    pro-n                                = prp-n.(dt-cle, dt-pre[pri-1/pro-1, ..., (pri-n/pro-n), dt-cov]
    ot-env = (dt-cle, dt-pre[pri-1/pro-1, ..., (pri-n/pro-n)], dt-cov) open-time environment
  true → (ot-env, dt-sto)

```

This constructor given an empty tuple of arguments returns a state-to-state function `open-pro-den`. This function gets a *declaration-time state* `dt-sta`, and generates a tuple of procedures as a least solution of a set of fixed-point equations that refer to pre-procedures declared in the classes of `dt-sta`. These procedures are then assigned to the

corresponding procedure indicators in the declaration-time environment, thus creating an *open-time environment*. This environment together with the declaration-time store, creates an *open-time state*.

Note that if we execute a “main program”, i.e., a program which is not a procedure body in a procedure call, then the declaration-time procedure environment **dt-pre** is empty.

It is worth observing in this place that in our definition we do not build any stack mechanism usually engaged associated with recursion by interpreters or compilers. We do not need to do so, since we describe the recursion in **Lingua** using the recursion in **MetaSoft**.

Now, let’s try to make the definition of our constructor a little more formal. To do this we first define a family of metaconstructors **MC[n]** indexed by positive integers  $n$ :

$$\begin{aligned} \text{MC}[n] &: (\text{ProInd} \times \text{PrePro})^{\text{cn}} \times \text{State} \mapsto \text{Procedure}^{\text{cn}} \mapsto \text{Procedure}^{\text{cn}} \\ \text{MC}[n].(((\text{pri-1}, \text{prp-1}), \dots, (\text{pri-n}, \text{prp-n})), \text{sta}).(\text{pro-1}, \dots, \text{pro-n}) = \\ &\quad \text{let} \\ &\quad ((\text{cle}, \text{pre}, \text{cov}), \text{sto}) = \text{sta} \\ &\quad \text{new-pro-1} = \text{prp-1}.\text{dt-cle}, \text{dt-pre}[\text{pri-1}/\text{pro-1}, \dots, (\text{pri-n}/\text{pro-n})] \\ &\quad \dots \\ &\quad \text{new-pro-n} = \text{prp-n}.\text{dt-cle}, \text{dt-pre}[\text{pri-1}/\text{pro-1}, \dots, (\text{pri-n}/\text{pro-n})] \\ &\quad \text{true} \rightarrow (\text{new-pro-1}, \dots, \text{new-pro-n}) \end{aligned}$$

Then by

$$\text{LFP} : (A \mapsto A) \mapsto A$$

we denote a universal function such that if  $A$  is a CPO (Sec. 2.4), and if  $F : A \mapsto A$  is a continuous function in this CPO, then  $\text{LFP}.F$  is the least fixed point of  $F$ . With these metafunctions we can write our definition in the following way:

$$\begin{aligned} \text{create-pro-opening} &: \mapsto \text{DecDen} \\ \text{create-pro-opening} &: \mapsto \text{WfState} \mapsto \text{WfState} \\ \text{create-pro-opening}().\text{dt-sta} &= \text{dt-sta a declaration-time state} \\ \text{is-error}.\text{dt-sta} &\rightarrow \text{dt-sta} \\ \text{get-pre-pro}.\text{dt-sta} = () &\rightarrow \text{dt-sta} \\ \text{let} \\ &((\text{dt-cle}, \text{dt-pre}), \text{dt-sto}) = \text{dt-sta} \\ &((\text{pri-1}, \text{prp-1}), \dots, (\text{pri-n}, \text{prp-n})) = \text{get-pre-pro}.\text{dt-sta} \\ &(\text{pro-1}, \dots, \text{pro-n}) = \text{LFP}.\text{MC}[n].(((\text{pri-1}, \text{prp-1}), \dots, (\text{pri-n}, \text{prp-n}))) \\ \text{true} &\rightarrow ((\text{dt-cle}, \text{dt-pre}[\text{pri-1}/\text{pro-1}, \dots, \text{pri-n}/\text{pro-n}]), \text{sto}) \end{aligned}$$

Of course,  $\text{Procedure}^{\text{cn}}$  is a CPO with a componentwise ordering and **Procedure** is ordered by a set-theoretical inclusion of functions. It remains to be proved that

$$\text{MC}[n].(((\text{pri-1}, \text{prp-1}), \dots, (\text{pri-n}, \text{prp-n})), \text{sta}))$$

is a continuous function in  $\text{Procedure}^{\text{cn}}$ .

## 7 SYNTAX AND SEMANTICS

### 7.1 An overview of syntax derivation

The derivation of a syntax in our model starts from the signature  $\text{SigAlgDen}^{51}$  of the algebra of denotations and proceeds in three steps corresponding to three transformations:

|                                   |                              |  |
|-----------------------------------|------------------------------|--|
| $\text{S2A} : \text{SigAlgDen}$   | $\mapsto \mathbf{AlgAbsSyn}$ | the creation of an algebra of abstract syntax <sup>52</sup>  |
| $\text{A2C} : \mathbf{AlgAbsSyn}$ | $\mapsto \mathbf{AlgConSyn}$ | the transformation of abstract syntax into concrete syntax   |
| $\text{C2C} : \mathbf{AlgConSyn}$ | $\mapsto \mathbf{AlgColSyn}$ | the transformation of concrete syntax into colloquial syntax |

Each of these transformations is a many-sorted function, and A2C is (additionally) a homomorphism. We build our algebras in such a way that for each of them there exists a corresponding many-sorted function of semantics:

|                                   |                           |                      |
|-----------------------------------|---------------------------|----------------------|
| $\text{A2D} : \mathbf{AlgAbsSyn}$ | $\mapsto \mathbf{AlgDen}$ | abstract semantics   |
| $\text{C2D} : \mathbf{AlgConSyn}$ | $\mapsto \mathbf{AlgDen}$ | concrete semantics   |
| $\text{SEM} : \mathbf{AlgColSyn}$ | $\mapsto \mathbf{AlgDen}$ | colloquial semantics |

The first two semantics are homomorphisms, i.e., are denotational semantics, whereas the third one is not. Since colloquial syntax will be the ultimate user-syntax of our language, its semantics will be called *the semantics of Lingua*.

All syntactic algebras will be described by corresponding equational grammars (Sec. 2.15). These grammars explicitly define the carriers of our algebras, and implicitly — i.e., by grammatical clauses — their constructors. For instance, the following grammatical equation:

$$\text{cre} : \text{ConRefExp} = \begin{array}{l} \text{ref (Identifier)} \\ \text{ref ConValExp at Identifier fer} \end{array} \quad |$$

defines the carrier of concrete reference expressions, and each line of this equation below the sign  $=$ , called a *grammatical clause*, defines a corresponding constructor of the algebra of concrete syntax:

$$\begin{array}{l} \text{con-ref-variable.ide} = \text{ref (ide)} \\ \text{con-ref-attribute.(cve, ide)} = \text{ref cve at ide fer} \end{array}$$

Abstract-syntax grammar is always LL(k) (see Sec. 2.16) which makes abstract-syntax programs easily parsable<sup>53</sup>. The derivation of this grammar can be made algorithmic.

Since abstract syntax is usually awkward to use, in the next step we build an *algebra of concrete syntax* which is, by the rule, a homomorphic image of the abstract-syntax algebra:

$$\text{A2C} : \mathbf{AlgAbsSyn} \mapsto \mathbf{AlgConSyn}$$

This step is not algorithmic, since here we take major decisions about the future shape of our syntax, and we try to make it possibly user friendly. In building concrete syntax, we must ensure that the corresponding concrete semantics exists. For this to be the case, A2C must be adequate (Sec. 2.14), which means that it must not glue more than A2D. If it is so, the concrete semantics is unique and satisfies the equation

<sup>51</sup> This metavariable is not typeset in bold since a signature of an algebra is not an algebra itself.

<sup>52</sup> S2A is read as “signature to abstract”, and the remaining symbols are read analogously.

<sup>53</sup> In fact, abstract syntax scripts may be regarded as linear representations of parsing trees.

$$C2D = A2C^{-1} \bullet A2D$$

where  $A2C^{-1}$  denotes a chosen inverse of  $A2C$ , and corresponds to a parsing step from concrete to abstract syntax. Of course, if  $A2C$  is not an isomorphism, then there is more than one parsing procedures “reversing”  $A2C$ , and therefore  $A2C^{-1}$  should be regarded as just one of them.

Although the majority of grammatical clauses of concrete syntax can be made user-friendly, a few of them may require further modifications. As a rule these modifications are not homomorphic, since, if they were, they could have been included in  $A2C$ . These modifications lead us to a colloquial-syntax grammar, and to the corresponding algebra. In this case we make sure that there exists a many-sorted function

$$RES : \mathbf{AlgColSyn} \mapsto \mathbf{AlgConSyn}$$

that we call a *restoring transformation*. It restores colloquial syntax “back to” the concrete one. Now, the semantics of our language may be regarded as a composition of two many-sorted functions:

$$SEM = RES \bullet C2D$$

This semantics, is no more denotational, since it is not a homomorphism. Still, as we are going to see in Sec. 7.5.1, it may be said to be “denotational to a large extent”.

In the following sections we describe equational grammars of our three syntaxes. For the sake of brevity we shall not list all clauses of these grammars, but only their typical examples. The used notation has been described in Sec. 2.15. The components of our many-sorted functions will be indexed by suffixes indicating the corresponding carriers of algebras. E.g.

$$A2C.ins : AbsIns \mapsto ConIns$$

is a component of  $A2C$  that corresponds to instructions.

## 7.2 Abstract syntax

### 7.2.1 General remarks

As a rule abstract syntax is a prefixed syntax which means that each syntactic element starts, with three exceptions, from a prefix that is (not quite formally) an *Arial Narrow* copy of an *Arial* metaname of the corresponding constructor of denotations.

### 7.2.2 Identifiers, class indicators and privacy statuses

In this category we have three grammatical equations (see Sec. 6.2):

$$\begin{array}{ll} \text{id} & : \text{Identifier} = \dots \\ \text{cli} & : \text{ClInd} = \text{empty-class} \mid \text{Identifier} \\ \text{pst} & : \text{PriSta} = \text{private} \mid \text{public} \end{array}$$

The names of these domains constitute the three exceptions mentioned in Sec. 7.2.1. We do not prefix them since they will be shared by all our three syntaxes: abstract, concrete and colloquial.

### 7.2.3 Type expressions

$$\begin{array}{ll} \text{ate} & : \text{AbsTypExp} = \\ & \quad \text{ted-create-bo}() \mid \text{ted-create-in}() \dots \quad | \\ & \quad \text{ted-constant}(\text{Identifier}, \text{Identifier}) \quad | \\ & \quad \text{ted-create-li}(\text{AbsTypExp}) \quad | \\ & \quad \dots \end{array}$$

## 7.2.4 Value expressions

We assume to be given (i.e. somehow defined as parameters of our model) four auxiliary syntactic domains. We do not call them “abstract syntactic”, since they will be common for all three syntaxes.

```
Identifier    = ...
BooleanSyn   = {true, false}
IntegerSyn   = ...
RealSyn      = ...
TextSyn      = ...
```

The elements of these domains are symbols or strings of symbols representing corresponding elements of denotational domains. Examples of syntactic representations of integers or reals may be: `432894713984713847` for an integer or `9874,0951208515584958490` for a real number. The grammatical equation corresponding to value expressions is the following:

```
ave : AbsValExp =
  ved-bo(BooleanSyn)           |
  ved-in(IntegerSyn)           |
  ved-re(RealSyn)              |
  ved-te(TextSyn)              |
  ved-variable(Identifier)     |
  ved-attribute(AbsValExp , Identifier) |
  ved-call-fun-pro(Identifier, Identifier, ActParAbs) |
  ved-divide-re(AbsValExp , AbsValExp) |
  ved-equal(AbsValExp , AbsValExp) |
  ved-or(AbsValExp , AbsValExp) |
  ved-create-li(AbsValExp)     |
  ved-get-from-rc(AbsValExp , Identifier) |
  ...
```

## 7.2.5 Reference expressions

```
are : AbsRefExp =
  ref-variable(Identifier)           | a reference of a variable
  ref-attribute(AbsValExp , Identifier) | a reference of an object attribute
```

## 7.2.6 Instructions

```
ain : AbsIns =
  skip-ins() |
  assign(AbsRefExp , AbsValExp) |
  call-imp-pro(Identifier , Identifier , ActParAbs , ActParAbs) |
  call-obj-con(Identifier , Identifier , ActParAbs) |
  if(AbsValExp , AbsIns , AbsIns) |
  if-error(AbsValExp , AbsIns) |
  while(AbsValExp , AbsIns) |
  compose-ins(AbsIns , AbsIns)
```

## 7.2.7 Declarations

```
ade : AbsDec =
  var-dec(AbsListOfIde , AbsTypExp) |
  enrich-cov-rel(AbsTypExp , AbsTypExp) |
  cla-dec(Identifier , AbsClaExp , AbsClaTra) |
```



```
compose-dec(AbsDec , AbsDec)
skip-dec()
```

## 7.2.8 Openings of procedures

```
aop : AbsOpePro = create-open-pro()
```

## 7.2.9 Class transformers

```
act : AbsClaTra =
  add-abs-att(Identifier , AbsTypExp, PriSta)
  ...
  add-con-imp-met(Identifier , AbsImpProSig , AbsPro)
  add-con-fun-met(Identifier , AbsFunProSig , AbsPro, AbsValExp)
  add-con-obj-con(Identifier , AbsObjConSig , AbsPro)
  concretize-fun-met(Identifier, AbsFunProSig, AbsPro, AbsValExp)
  ...
  compose-cla-tra(AbsClaTra, AbsClaTra)
  skip-ctr()
```

## 7.2.10 Preambles of programs

```
app : AbsProPre =
  make-ppd-of-dcd(AbsDec)
  make-ppd-of-ind(AbsIns)
  compose-ppd(AbsProPre , AbsProPre)
```

## 7.2.11 Programs

```
apr : AbsPro = make-prog-den( AbsProPre , AbsOpePro , AbsIns )
```

## 7.2.12 Declaration-oriented carriers

```
ali : AbsLisOfIde =
  build-loi(Identifier)
  add-to-loi(Identifier , ListOfIde)

ads : AbsDecSec =
  build-dse(AbsLisOfIde , AbsTypExp)

afp : AbsForPar =
  build-fpd(AbsDecSec)
  add-to-fpd(AbsDecSec , AbsForPar)

aap : AbsActPar =
  build-apd(AbsLisOfIde)
```

Intuitively the last equation means that abstract actual parameters are just list of identifiers. Set-theoretically we could have dropped the category `AbsActPar`, and use `AbsLisOfIde` instead, but algebraically we keep it formally have the concept of actual parameters in our model.

### 7.2.13 Signatures

```

ais  : AbsImpProSig =
      build-ipsd(AbsForPar , AbsForPar)

afs  : AbsFunProSig =
      build-fpsd(AbsForPar , AbsTypExp)

aos  : AbsObjConSig =
      build-ocsd(AbsForPar , Identifier)

```

## 7.3 Concrete syntax

### 7.3.1 General remarks

In the abstract-to-concrete step, we modify syntax to make it more user-friendly but keep its semantics denotational. Technically, our modifications will belong to four categories:

1. the simplification of prefixes,
2. the omission of prefixes but keeping parenthesizing,
3. the modifications from prefix to infix notation,
4. the omission of parentheses wherever possible without violating semantics' homomorphicity.

The first three categories include transformations that are isomorphic-like, i.e. unambiguously reversible. The fourth group makes A2C not isomorphic, but still homomorphic. In this step, we go only as far, as the adequacy of A2C permits (cf. Sec. 2.14), i.e. we omit parentheses corresponding to sequential composition of declarations and instructions, but not to arithmetic operations. In this step we sacrifice LL(k)-ness for friendliness.

In the end we have to emphasize that our discussion of the abstract-to-concrete step is very sketchy. We only try to show a general potential of this step, leaving its further elaboration and analysis to a future research.

### 7.3.2 Identifiers, class indicators and privacy statuses

```

ide  : Identifier = ...
cli  : ClaInd   = empty-class | Identifier
pst  : PriSta   = private | public

```

### 7.3.3 Type expressions

```

cte  : ConTypExp =
      boolean | integer ... |
      Identifier . Identifier ) |
      list-of(ConTypExp ) |
      ...

```

In two first clauses we only shorten prefixes, which is an isomorphic-like transformation. In the second clause we allow writing `MyClass.myType` instead of `ted-constant(MyClass, myType)` which is again an isomorphic transformation, although now the proof of this fact may be not trivial. Note that a script of the form `ide-1.ide-2` may be a type constant or a value expression, and we have to see the context of this script to identify which one it is.

### 7.3.4 Value expressions

We assume to be given the same auxiliary domains as in Sec. 7.2.3. The grammatical equation defining concrete value expressions is the following:

```

cve : ConValExp =

```

|  |  |   |
|--|--|---|
| <code>true   false</code>                          |  | constant value expressions                              |
| <code>IntegerSyn</code>                            |  | syntactic representations of integers                   |
| <code>RealSyn</code>                               |  | syntactic representations of reals                      |
| <code>' TextSyn '</code>                           |  | texts are closed in apostrophes                         |
| <code>Identifier</code>                            |  | variable  |
| <code>ConValExp . Identifier</code>                |  | getting an attribute of an object                       |
| <code>call Identifier.Identifier(ConActPar)</code> |  | calling an imperative procedure                         |
| <code>(ConValExp /. ConValExp)</code>              |  | real division is a “division with dot <code>/.</code> ” |
| <code>(ConValExp = ConValExp)</code>               |  | equality  |
| <code>(ConValExp or ConValExp)</code>              |  | disjunction   |
| <code>make_list(ConValExp)</code>                  |  | making a one-element list                               |
| <code>rec ConValExp at Identifier cer</code>       |  | getting an attribute of a record                        |
| ...  |  |   |

Here we switch from prefix- to infix notation, but we keep the parentheses structures, although the symbols of parentheses may change, e.g. from “mathematical” ones like `(` and `)` to program oriented like `rec` and `cer`. An example of a clause of the definition of `A2C.cve` may be:

```
A2C.cve.[ ved-call-fun-pro(aid-c , aid-p , apa) ] = -c for “class, -p for “procedure”
  call A2C.cid.[aid-c] . A2C.cid.[aid-p] ( A2C.apa.[apa] )
```

### 7.3.5 Reference expressions

```
cre : ConRefExp =
  ref (Identifier) |
  ref ConValExp at Identifier fer
```

### 7.3.6 Instructions

```
cin : ConIns =
  skip-ins |
  ConRefExp := ConValExp |
  call Identifier.Identifier ( val ConActPar ref ConActPar ) | calls of object constr.
  new Identifier by Identifier.Identifier (ConActPar) |
  if ConValExp then ConIns else ConIns fi |
  if error ConValExp then ConIns fi |
  while ConValExp do ConIns od |
  ConIns ; ConIns |
```

In the first clause we changed prefix notation to infix notation, and we skipped parentheses. The latter transformation is not harmful for the adequacy of the homomorphism, since — intuitively — assignment instructions will be always closed by the parentheses of “other structures” such as `;`, `do`, `then`, etc. A formal proof should be carried by induction on the structure of our grammar.

In the last clause we have also skipped parentheses, but in this case to prove that such a transformation does not destroy the adequacy of our homomorphism we have to use the fact that the sequential composition of functions is associative, and refer to Theorem 2.14-1 in Sec 2.14.

### 7.3.7 Declarations

```
cde : ConDec =
  let ConLisOfIde be ConTypExp tel |
  enrich-cov( ConTypExp, ConTypExp ) |
  class Identifier parent ConClaExp with ConClaTra ssalc |
  skip-dec |
```

ConDec ; ConDec

Similar comments, as in Sec. 7.3.6, apply here to the last clause.

### 7.3.8 Openings of procedures

cpo : ConProOpe = **open procedures**

### 7.3.9 Class transformers

```

cct : ConClaTra =
  let Identifier be ConTypExp as PriSta tel           | add abs. attribute
  ...
  proc Identifier (ConImpProSig) begin ConPro end      |
  fun Identifier (ConFunProSig) begin ConPro return ConValExp end |
  obj Identifier (ConObjConSig) begin ConPro end       |
  con-fun Identifier (ConFunProSig) begin ConPro return ConValExp end |
  ...
  ConClaTra ; ConClaTra                               |
  skip-ctr

```

The following component of procedure declaration:

Identifier (ConImpProSig)

will be called a *procedure header*.

### 7.3.10 Preambles of programs

```

cpp ConProPre =
  ConDec           |
  ConIns           |
  ConProPre ; ConProPre

```

### 7.3.11 Programs

cpr : ConPro = ConProPre ; **open procedures** ; ConIns

Here we may safely drop parentheses since they are the outermost parentheses in a program, and therefore their removal do not destroy isomorphism.

### 7.3.12 Declaration-oriented carriers

```

cli : ConLisOfIde =
  Identifier           |
  Identifier , ConLisOfIde

cds : ConDecSec =
  ConLisOfIde as ConTypExp

cfp : ConForPar =
  ConDecSec           |
  ConDecSec , ConForPar

cap : ConActPar =
  ConLisOfIde

```

See a comment about the last equation in Sec. 7.2.12.

### 7.3.13 Signatures

```

cis  : ConImpProSig  = val ConForPar ref ConForPar
cfs  : ConFunProSig  = val ConForPar ret ConTypExp
col  : ConObjConSig  = val ConForPar ret Identifier

```

An example of a concrete imperative-procedure declaration may be the following

```
proc compute (val x, y as integer, z as real, ref p, r as integer )
begin
    cpr
end
```

where `cpr` is a concrete program.

## 7.4 Colloquial syntax

Colloquial syntax is, as a rule, THE syntax of our language, i.e., the syntax to be used by programmers. Consequently the metavariables (non-terminals) of colloquial syntax will not be prefixed by **Col**. E.g., instead of writing **ColValExp** we shall write just **ValExp**.

In our language we introduce two categories of colloquialisms: the omission of parentheses in arithmetic and boolean expression, and the creation of a new constructor of attribute declarations.

The omission of parentheses in arithmetic and boolean expressions concerns value expressions. Formally this means that in a concrete-to-colloquial step to every parenthesized colloquial clause such as, e.g.,

$$(\text{ValExp} +, \text{ValExp})$$

we add a corresponding parentheses-free clause

ValExp +, ValExp

Consequently the grammatical equation for colloquial value-expressions will look as follows:

```
vex : ValExp =
  true | false
  IntegerSyn
  RealSyn
  ' TextSyn '
  Identifier
  obj ValExp at Identifier jbo
  call Identifier.Identifier(ActPar)
  (ValExp /. ValExp)
  ValExp /. ValExp
```

new clause (without parentheses)

In colloquial syntax parentheses are optional — we may use them or not. The signature of the algebra of colloquial syntax is, therefore, an extension of the signature of concrete syntax by constructors building parentheses-free expressions.

The corresponding restoring transformation for arithmetic expressions adds parentheses according to the rule that multiplication and division bind stronger than addition and subtraction, and the “remaining” parentheses are added from left to write. E.g., the colloquial expression:

$$a + b * c - e * f$$

will be restored to

$((a + (b * c)) - (e * f)).$

## 7.4.1 The list of colloquial domains

Since we are going to use our colloquial syntax in the investigations about validating programming in Sec. 9, we list below all colloquial-syntax domains and their corresponding metavariables.

|     |               |                                   |
|-----|---------------|-----------------------------------|
| ide | : Identifier  | — identifiers                     |
| cli | : Clalnd      | — class indicators                |
| pst | : PriSta      | — privacy statuses                |
| tex | : TypExp      | — type expressions                |
| vex | : ValExp      | — value expressions               |
| rex | : RefExp      | — reference expressions           |
| ins | : Instruction | — instructions                    |
| dec | : Declaration | — declarations                    |
| opp | : OpePro      | — the opening of procedures       |
| ctr | : ClaTra      | — class transformers              |
| ppr | : ProPre      | — program preambles               |
| prg | : Program     | — programs                        |
| loi | : LisOfIde    | — lists of identifiers            |
| des | : DecSec      | — declaration sections            |
| fpa | : ForPar      | — formal parameters               |
| apa | : ActPar      | — actual parameters               |
| ips | : ImpProSig   | — imperative-procedure signatures |
| fps | : FunProSig   | — functional-procedure signatures |
| ocs | : ObjConSig   | — object-constructor signatures   |

## 7.5 Semantics

### 7.5.1 The ultimate semantics of Lingua

Since in our model colloquial syntax is assumed to be the user's "ultimate" syntax, its semantics:

**SEM** : **AlgColSyn**  $\mapsto$  **AlgDen**

will be called just *the semantics* of **Lingua**. It is not a homomorphism, and symbolically may be written as a composition of three many-sorted mappings

$$\text{SEM} = \text{RES} \bullet \text{A2C}^{-1} \bullet \text{A2D}$$

where:

- **RES** is a restoring transformation from colloquial syntax to concrete syntax,
- **A2C<sup>-1</sup>** is a chosen invers of **A2C** and represents a parsing step,
- **A2D** is a homomorphism which constitutes the semantics of abstract syntax.

It is to be emphasized in this place that whereas the choice of **A2C<sup>-1</sup>** is irrelevant for **SEM**, the choice of **RES** defines the way in which we understand colloquial syntax.

Let us start the process of building the definition of semantics from the definition of **A2D**. This is an easy step, since the grammar of abstract syntax is unambiguous. In this step for each syntactic category, i.e., for each carrier of the algebra of abstract syntax, we create one definitional equation with several *semantical clauses*. E.g., for the category of programs the equation includes only one clause, and is the following:

**A2D**apr.[apr] =  
 apr :: **make-prog**( ade , **create-pro-opening**() , ain )  $\rightarrow$   
 make-prog.(**A2D**.ade.[ade], **create-pro-opening**(), **A2D**.ain.[ain])



In the first line of this definition, the first **apr** is an index, and the **apr** in square brackets is a metavariable running over **AbsPro**.

The second and the third line constitute together one semantical clause. The symbol  $::$  denotes (not quite formally) a pattern matching operator, and expresses the fact that **apr** is parsable to the form **make-prog**(**ade** , **create-pro-opening**() , **ain** ). Since our grammar is unambiguous, this parsing is unique.

The constructor of denotations **make-prog** is the semantic counterpart of the prefix **make-prog**. This constructor is applied to the denotations of the components of the program, which expresses the compositionality (denotationality) of the semantics or abstract syntax. Another typical example, which this time concerns instructions, is the following:

```
A2Dain.[ain] =
  ain :: assign(are , ave)           →
    assign.(A2Dare.[are], A2Dave.[ave])
  ain :: call-imp-pro(aid-1.aid-2 , apa-1 , apa-2) →
    call-imp-pro.( A2Daid.[aid-1] , A2Daid.[aid-2] , A2Dapa.[apa-1] , A2Dapa.[apa-2])
  ...
  ain :: seq-ins(ain-1 , ain-2)      →
    seq-ins.(A2Dain.[ain-1], A2Dain.[ain-2])
```

Now, let's pass to the definition of  $A2C^{-1}$ . Since  $A2C$  is not an isomorphism, we have to choose one of alternative parsing strategies, but since it is adequate, this choice is irrelevant for the “meaning” of  $C2D$ . Note also that  $A2C$  introduces only “three ambiguities”, namely the omissions of parentheses in three grammatical concrete clauses:

```
(ConIns ; ConIns)
(ConDec ; ConDec)
(ConClaTra ; ConClaTra)
```

Since these cases are similar to each other, let's analyze the case of instructions. To define a chosen parsing strategy — let's call it  $C2A$  — we introduce an auxiliary subdomain of instructions called *atomic instructions*:

```
atin : AtomIns =
  ConRefExp := ConValExp
  call Identifier.Identifier ( val ConActPar ref ConActPar )
  new Identifier by Identifier.Identifier (ConActPar)
  while ConValExp do ConIns od
  if ConValExp then ConIns else ConIns fi
  skip-ins
```

It is now easy to prove (by induction) that every concrete non-atomic instruction is unambiguously parsable into an instruction of the form:

```
atin ; cin,
```

where **cin** may, of course, include some nonatomic instructions.

We emphasize that **AtomIns** is not regarded as a new carrier of the algebra of concrete syntax, but as an auxiliary domain “outside” of this algebra. We may say that we do not modify the algebra of concrete syntax, but we build an auxiliary one, to be used only for the purpose of parsing<sup>54</sup>.

Now, the definition of  $C2A.cin$  may be written as follows

```
C2Acin.[cin] =
  cin :: cre := cve           →
```

<sup>54</sup> Of course, we could have introduced an analogous construction already on the level of the algebra of denotations. We didn't do so, since in our method, we want to sharply distinguish between the stages of building denotations and of building syntax. In our opinion a language designer should not think of syntax, when designing the core of the language represented by denotations.

```

    assign(C2Acre.[cre] , C2Acve.[cve])
cin :: call cid-c.cid-p ( val cap-v ref cap-r)  →
    call-imp-pro(C2Acid.[cid-c] , C2Acid.[cid-p] , C2Acap.[cap-v] , C2Acap.[cap-r])
...
cin :: atin ; cin                               → seq-ins(C2Acin.[atin] , C2Acin.[cin])

```

From this definition, the definition of A2D, and the equation

$$C2D = C2A \bullet A2D \quad (7.5-1)$$

we can algorithmically generate the following equation of the definition of C2D:

$$\begin{aligned}
 C2Dcin.[cin] = & \quad (7.5-2) \\
 cin :: cre := cve & \quad \rightarrow \\
 \quad assign.(C2Dcre.[cre], C2Dcve.[cve]) & \\
 cin :: call cid-1.cid-2 ( val cap-1 ref cap-2) & \rightarrow \\
 \quad call-imp-pro.(C2Dcid.[cid-1] , C2Dcid.[cid-2] , C2Dcap.[cap-1] , A2Dcap.[cap-2]) & \\
 ... & \\
 cin :: atin ; cin & \rightarrow \\
 \quad seq-ins.(C2Dcin.[atin], C2Dcin.[cin]) &
 \end{aligned}$$

Let's see how it works for the first clause:

$$\begin{aligned}
 C2Dcin.[cre := cve] & = & \text{by (7.5-1)} \\
 A2Dain.[C2Aain.[ cre := cve ]] & = & \text{by isomorphicity of C2A} \\
 A2Dain.[ assign(C2Acre.[cre] , C2Acve.[cve]) ] & = & \text{by homomorphicity of A2D} \\
 assign.( A2Dare.[ C2Acre.[cre] ] , A2Dave.[C2Acve.[cve] ] ) & = & \text{by (7.5-1)} \\
 assign.( C2Dcre.[cre], C2Dcve.[cve] ) &
 \end{aligned}$$

In the second transformation we use the fact that in the case of assignments, A2Cain hence also C2Acin, are reversible, i.e. “locally isomorphic”.

Let us pass now to the restoring function RES. Similarly as C2A, also RES adds parentheses, but now, the way it does it makes difference for the meaning of expressions. E.g., if we decide to add the “missing” parentheses to

$$a + b * c + (d - e) * f$$

in assuming that multiplication bind stronger than addition, and besides we add them from left to right:

$$( (a + (b * c)) + ((d - e) * f) )$$

then we decide not only about the functioning of a parser, but also about the meaning of the expression.

We shall not go into the technical details of a definition of RES assuming that it has been somehow (chosen and) defined. Let us think, therefore, about the definition of SEM, which is a composition of RES with the semantics of concrete syntax:

$$SEM = RES \bullet C2D$$

The definition of this semantics may be created algorithmically from the definition of C2D. Let's show it on the example of equation (7.5-2). Its colloquial counterpart will be the following:

$$\begin{aligned}
 SEMins.[ins] = & \\
 RES.ins :: cre := cve & \rightarrow \\
 \quad assign.(C2Dcre.[cre], C2Dcve.[cve]) & \\
 RES.ins :: call cid-1.cid-2 ( val cap-1 ref cap-2) & \rightarrow \\
 \quad call-imp-pro.(C2Dcid.[cid-1] , C2Dcid.[cid-2] , C2Dcap.[cap-1] , A2Dcap.[cap-2]) & \\
 ... & \\
 RES.ins :: atin ; cin & \rightarrow \\
 \quad seq-ins.(C2Dcin.[atin], C2Dcin[cin]) &
 \end{aligned}$$

In this definition we first restore a colloquial instruction `ins` into a corresponding concrete one `RES.ins`, and then we parse it to apply `C2D cin`. Note that `C2Dcin` also adds some parentheses (to instructions) by using `C2A`.

## 7.5.2 Why do we need a denotational semantics?

A denotational semantics of a programming languages constitutes a fundament for the realization of at least three goals:

1. to build an implementations of the language, i.e. an interpreter or compiler,
2. to write a concise, complete and consistent user manual,
3. to establish constructors of functionally correct programs.

Regarding the first goal, the definitional clauses of a denotational semantics may be regarded as procedures of an interpreter. They mutually call themselves, and call also constructors of denotation. As such they should be easily implementable. They also indicate a systematic way to the development of a compiler.

A denotational semantics is also an adequate starting point for writing a user manual. Even if a user is not prepared to read, and understand denotational equations, these equations constitute guidelines for an author of a manual. Translated into intuitive explanations — as we did in Sec. 6 — result with a manual that is consistent, complete and concise at the same time. An experiment of writing a manual in this way has been described in .

One of the well-known nightmares of manual readers is that manual usually don't keep up with the updates of implementations. The existence of a mathematical semantics of a language which should be conformant with both, the implementation and the manual, helps in keeping the adequacy of manuals.

It is also to be mentions in this place that although a denotational semantics needs not be a core of a manual, it should be contained in it as a standard to be referred to in cases of doubts. In such cases it may be practical to write semantic clauses in an unfolded form. E.g. instead or writing:

```
SEM.ins.[rex := vex] =
  assign.(SEM.rex.[rex], SEM.vex.[vex])
```

we may prefer to write

```
SEM.ins.[rex := vex].sta =
  is-error.sta           → error.sta
  let
    red  = SEM.rex.[rex]
    ved  = SEM.vex.[vex]
  ved.sta = ?           → ?
  ved.sta : Error       → sta ◀ ved.sta
  red.sta  : Error      → sta ◀ red.sta
  let
    val                = ved.sta
    ref                = red.sta
    (tok, (typ, re-ota)) = ref
    (env, (obn, dep, st-ota, sft, 'OK')) = sta
  re-ota ≠ $ and re-ota ≠ st-ota → 'reference not visible'
  not ref VRA.cov val          → 'incompatibility of types'
  true                        → (env, (obn, dep[ref/val], cov, st-ota, sft, 'OK'))
```

Whereas implementations and manuals may be created without a mathematical semantics — which, unfortunately, is a fairly common practice — it is hard to expect that we could create (mathematically) sound program construction rules without it. How to do when we have such semantics, we explain in Sec. 9.

Regarding writing manuals based on denotational semantics, this work is in progress.

## 8 RELATIONAL MODEL OF PROGRAM CORRECTNESS

### 8.1 Historical remarks

Semantic correctness of programs, historically called *program correctness*, was a subject of investigations from the very beginning of the computer era. The earliest paper in this field — today practically forgotten — has been published by a British mathematician, Alan Turing<sup>55</sup>, in 1949 [93]. Nearly twenty years later, in 1967, the same ideas were investigated again by an American scientist, Richard Floyd [53]. In 1978, the Association for Computing Machinery established the annual Turing Price “for outstanding achievements in informatics”. One of the first winners of that price in 1978 was... Richard Floyd.

As far as we know, it has never been officially established if Floyd knew Turing’s work. In the 1980-ties, A. Blikle wrote to Cambridge University on that issue. The only answer he received was substantial advice: do not try to build “yet another myth about Turing”.

The work of Floyd introduced a fundamental concept of *an invariant of a program* and was dedicated to programs represented by graphical forms called *flow-diagrams* or *frow-charts*. In 1969, a British scientist, C.A.R. Hoare (also a Turing Price winner), published a paper [61] concerning Floyd’s ideas applied to *structural programs*, i.e., programs constructed with the help of three constructors: sequential composition, if-then-else branching, and while loops. The works of C.A.R. Hoare and his followers, called *Hoare’s logic*, were later summarized in two extensive monographs by K. Apt [4] and by K. Apt and H.R. Olderog [5].

The correctness of programs defined by C.A.R. Hoare was called *partial correctness*. A program is partially correct for a precondition *prc* and a postcondition *poc* if, whenever *prc* is satisfied by an input state, and the execution of this program terminates, then the terminal state satisfies *prc*.

Despite its undoubted scientific importance, the idea of proving programs correct was never widely applied in software industry. In our opinion, this situation was due to the implicit assumption that programs come first and their proofs are built later. This order, although natural in mathematics, where a theorem precedes its proof, is quite unusual in engineering. Imagine an engineer who first constructs a bridge and only later performs all the necessary calculations. Such a bridge would probably collapse before its construction was completed, and in fact, this is what unavoidably happens with programs. The first version of a code usually does not work as expected. Consequently, a large part of the program-development budget is spent on testing and “debugging”, i.e., on removing bugs introduced at the stage of writing the code. It is a well-known fact that all bugs can never be identified and removed by testing. Therefore, the remaining bugs are removed at the user’s expense under the name of “maintenance”. This process practically never terminates.

In this place, it is worth quoting a thought of Edsger W. Dijkstra that he called a “sad remark”<sup>56</sup>:

*Since then we have witnessed the proliferation of baroque, ill-defined and, therefore, unstable software systems. Instead of working with a formal tool, which their task requires, many programmers now live in a limbo of folklore, in a vague and slippery world, in which they are never quite sure what the system will do to their programs. Under such regretful circumstances the whole notion of a correct program — let alone a program that has been proved to be correct — becomes void. What the proliferation of such systems has done to the morale of the computing community is more than we can describe.*

Even though these words were written nearly half a century ago, and during this time, the reliability of hardware and the applicability of software has increased by several orders of magnitude, the problems pointed out by E.W. Dijkstra are still there.

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<sup>55</sup> Alan Turing (1912-1954) was one of the creators of the theory of computability. His model known today as *Turing machine* is regarded as one of fundamental concepts of this theory. Due to his work "On Computable Numbers, With an Application to the Entscheidungsproblem" Turing was considered as one of the greatest mathematicians of the world. Unfortunately he was also subject to a homophobic discrimination. When in 1952 police has learned about his homosexuality he was forced to choose between prison or hormonal therapy. He has chosen the latter but committed a suicide.

<sup>56</sup> In [51] published in 1976 page 202.

As a conclusion of his remark Dijkstra proposed an alternative approach to program correctness, first sketched in [50] and then developed in [51]. Two issues make Dijkstra's approach different from the Hoare's.

The first is about the very notion of program correctness. Hoare's partial correctness was strengthened to *total correctness* where the satisfaction of a precondition guarantee the termination of program execution and the satisfaction of a postcondition in the end. The difference concerns, therefore, what we prove about a program.

The second difference concerns the question of when program correctness is established. From this perspective Hoare's approach may be said to be *reactive* — we first write a program, and only later prove its correctness — whereas the approach of Dijkstra's is *proactive* — programs are developed in a way that guarantees their correctness.

Still there are two similarities of these approaches which make them different from our approach.

First, in both cases a nontermination of a program execution essentially means that the program loops indefinitely. No error messages are considered and consequently boolean expressions and conditions represent total two-valued function.

Second, the adequacy (soundness) of the rules of proving programs correct or building correct programs was assumed rather than proved. In other words both authors presupposed that if their methods were to be used in "practical situations", one had to prove that for the used programming language their rules were satisfied. Whereas for their (very simple) exemplary languages such assumption were fairly credible, in the case of "real programming languages", appropriate adequacy proofs may be far from obvious.

In this book we are developing a proactive approach initiated by A.Blikle in the papers [25], [26] and [27] published at the turn of the decades 1970s and 1980s. In our case:

- the (denotational) semantics of the exemplary programming language is defined explicitly, and the language is pretty closed to real programming languages,
- on the ground of this semantics program construction rules are proved to be sound,
- the nontermination of program execution may mean not only an indefinite execution, but also abortion signalized by an error message; consequently our concept of correctness is a *clean (non-abortion) total correctness* as define by A.Blikle in [28],
- to prove clean termination of programs we have to use three-valued boolean expressions and conditions,

In the end let us mention a few other approaches to program correctness developed in Poland in in the years 1970 – 1990. The first paper on that subject (and technically different from Hoare's) was published in 1971 by A. Mazurkiewicz [75]. A year later, during the first conference in a series called *Mathematical Foundations of Computer Science*<sup>57</sup>, A. Blikle and A. Mazurkiewicz presented a joint paper [37] on program correctness based on an algebra of binary relations and covering recursive programs with nondeterminism.

Another two approaches to program correctness were developed at Warsaw University. The first was a formalized approach called *algorithmic logic* [10], where logical formulas syntactically include programs. The second [73] was much more engineering-oriented and split into three areas: *grammatical deduction*, *performance analysis of computing systems*, and *formal specification of software requirements*. An interesting application of the second approach is described in a paper by D.L. Parnas, G.J.K. Asmis, and J. Madey [83] devoted to software safety assessment for a Darlington Nuclear Power Generating Station (Canada) shutdown system.

## 8.2 A relational model of nondeterministic programs' correctness

Since the rules for developing correct programs are derived from the rules of proving programs correct, we shall start with the latter. The discussion will be carried in an algebra of binary relations since this leads to a relatively simple model where many technicalities of programming languages can be hidden. Of course, to apply these rules

---

<sup>57</sup> This conference was organized in 1972 by a group of young researchers from the Institute of Computer Science of the Polish Academy of Sciences and the Department of Mathematics and Mechanics of Warsaw's University. Next year a similar conference was organized in Czechoslovakia which gave rise to a long series of MFCS conferences. Since 1974 proceedings of these conferences were published by Springer Verlag in the series Lecture Notes in Computer Science.



in a practical environment, they have to be expressed on the grounds of a mathematical model of a programming language. A language **Lingua-V** (V for “validation”) with such a model is described in Sec. 9.

Each program, and each of its imperative components, defines an *input-output relation* (I-O relation) between its input states, and the corresponding output states. Of course, in a deterministic case, this relation is a function. Although programs in **Lingua** are deterministic, the discussion of a (possibly) non-deterministic case seems worthwhile, especially since it does not complicate the model.

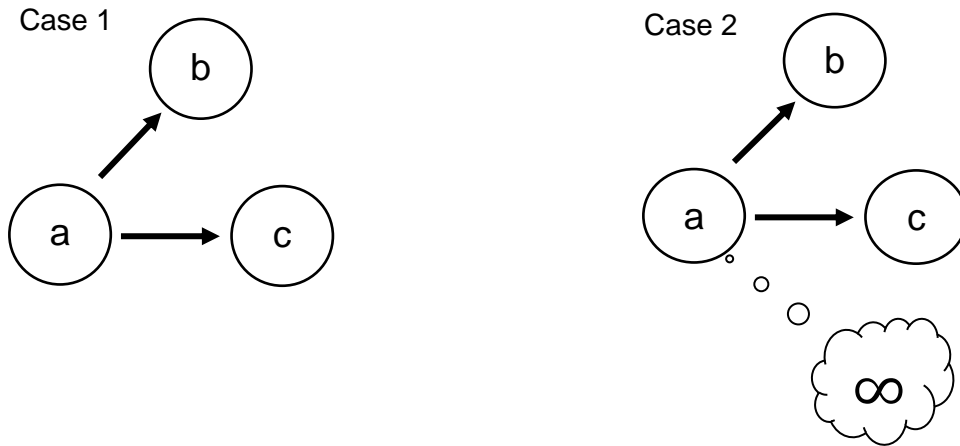
Let  $S$  be an arbitrary, possibly infinite, set of elements called *states*. In **Lingua**, states are fairly complex items but in the abstract case, we do not need to assume anything about them. In the relational model programs are represented by binary relations over  $S$ , i.e., elements of the set:

$$\text{Rel}(S, S) = \{R \mid R \subseteq S \times S\}$$

For short, we shall refer to them as just programs. This will avoid any confusion since in this section, we shall not talk about syntax; therefore, programs may be identified with their denotations. The fact that

$$a R b \quad \text{for } a, b : S$$

means that there exists an execution of program  $R$  that starts in  $a$  and terminates in  $b$ .



**Fig. 8.2-1 Two nondeterministic cases**

In a non-deterministic case, there may be more than one execution that starts in  $a$ . Some may terminate with another state, say  $c$  (Case 1 of Fig. 8.2-1), some others may be infinite (Case 2 of Fig. 8.2-1). In our model, the difference between Case 1 and Case 2 cannot be expressed. In both cases, we can only say that

$$a R b \text{ and } a R c.$$

Note that due to the use of states which may carry errors, abortion of a computation from  $a$  to  $b$  means that  $b$  carries an error. This also means that if  $R$  is a function then the non-existence of a state  $b$  such that  $a R b$  means that  $a$  starts an infinite execution.

If we want to deal with infinite executions explicitly, we need a different concept of program denotations. Two such models were analyzed in [23]. One uses so-called  $\delta$ -relations, where  $a R \delta$  means that there exists an infinite computation that starts in  $a$ <sup>58</sup>. In this model, however, we cannot describe the fact that there are two or more different infinite computations that start from the same state. Such issues can be handled on the ground of the second model, where program denotations are sets of finite or infinite sequences of states called *bundles of computations*. Both approaches can be used in building denotational models of programming languages.

<sup>58</sup> In this model each  $\delta$ -relation is a union of three set of pairs  $R \subseteq S \times S$ ,  $D \subseteq S \times \{\delta\}$  and  $\{(\delta, \delta)\}$ , where  $R$  and  $D$  may be empty.

### 8.3 Iterative programs

In “prehistoric” informatics of the years 1940/1950, programs were written as lists of labeled instructions executed sequentially one after another unless a *jump instruction* **goto** interrupted that flow. With jump instructions one can build an arbitrary graph of elementary instructions called a *flow-diagram*. Early papers on program correctness were devoted to such programs later called *iterative programs*.

A general relational model of an iterative program is the following fixed-point set of so called left-linear equations<sup>59</sup>:

$$\begin{aligned} X_1 &= R_{11} X_1 \mid \dots \mid R_{1n} X_n \mid E_{1n} \\ &\dots \\ X_n &= R_{n1} X_1 \mid \dots \mid R_{nn} X_n \mid E_{nn} \end{aligned} \tag{8.3-1}$$

that corresponds to a graph whose nodes are numbers  $1, \dots, n$ , each relation  $R_{ij}$  labels a unique edge between  $i$  and  $j$ , and each  $E_{in}$  (exit relation) is a “dangling edge” that start on  $i$ , but does not point to any other node. Note that in a nondeterministic case  $R$ ’s and  $E$ ’s represent bunches of “alternative computations”.

The code of such a program may be written as an arbitrarily ordered<sup>60</sup> sequences of labelled instructions of the form:

$i$  : **do**  $R_{ij}$  **goto**  $j$  and  
 $i$  : **do**  $E_{in}$ .

If there is no instruction between  $i$  and  $j$ , then the relation  $R_{ij}$  is empty which means that there are no executions between  $i$  and  $j$ . Since the atomic instructions  $R_{ij}$  and  $E_{in}$  are not necessarily functions, such a program may have a non-deterministic character. For (8.3-1) to be deterministic, two conditions must be satisfied:

- all  $R_{ij}$  and  $E_{in}$  must be functions,
- for every  $i$ , all  $R_{i1}, \dots, R_{in}$  and  $E_{in}$  must have disjoint domains.

As has been proved in [28], if  $(P_1, \dots, P_n)$  is the least solution of (8.3-1), then  $P_i$  is the input-output relation of the path from node  $i$  to node  $n$ . Therefore, if we assume that 1 represents the initial node, and  $n$  is the final node, then  $P_1$  is the input-output relation (the denotation) of our program. The class of iterative programs understood in that way, together with their correctness-proof rules, were investigated in [23] and [28]. It is worth mentioning in this place that  $P_i$ ’s correspond to A. Mazurkiewicz *tail functions* [75] published in 1971 and to Ch. Strachey and Ch. Wadsworth *continuations* [90] published in 1974.

Programmers of the decade 1950/1960 were competing with each other in building more and more complicated flowchart programs that usually nobody except them was able to understand. Unfortunately, quite frequently, the authors themselves were not able to predict the behavior of such programs.

As a reaction to these problems, first algorithmic programming languages such as Fortran and Algol-60 were created. They were offering tools for *structural programming* such as sequential composition, **if-then-else**, and **while**<sup>61</sup>. Such programs were much easier to understand and also allowed for significant simplification of program-correctness proof rules.

<sup>59</sup> They are called so because coefficients of variables  $X_i$  stand on their left-hand side. A symmetric model of right-linear equations of the general form  $X = XR \mid Q$  has been analyzed in [19].

<sup>60</sup> The execution of such a program does not depend on the order of its instructions since every instruction points to the instruction which should be executed as the next one.

<sup>61</sup> The author who introduced the term “structured programming” was a Dutch computer scientist Edsger Dijkstra (see [50] and [51]).



In the sequel, we shall restrict our discussion to only three primary *structural constructors* since they allow for the implementation of any “implementable” function<sup>62</sup>:

1. sequential constructor denoted by a semicolon “;”,
2. conditional constructor if-then-else-fi,
3. loop constructor while-do-od.

The sequential composition is the composition of relations (functions) as defined in Sec. 2.7. To define the remaining constructors, we have to introduce additional concepts. Assume that  $p$  is a 3-valued predicate, i.e.

$$p : S \mapsto \{tt, ff, ee\}$$

Since in our case the denotations of boolean expressions will be representable by such predicates, in the relational each of them will be represented by two disjoint set of states:

$$\begin{aligned} C &= \{s \mid p.s = tt\} \\ \neg C &= \{s \mid p.s = ff\} \end{aligned}$$

Of course, if  $p$  is a 2-valued, then  $C \mid \neg C = S$ , and therefore only one set is necessary to represent it. Notice also that our model does not distinguish between the two cases:

$$\begin{aligned} p.s &: \text{Error} \\ p.s &= ? \end{aligned}$$

In both of them  $s : S - (C \mid \neg C)$ . If we want to distinguish between these cases, we have to represent predicates by three disjoint sets:

$$\begin{aligned} C &= \{s \mid p.s = tt\} \\ \neg C &= \{s \mid p.s = ff\} \\ eC &= \{s \mid p.s : \text{Error}\} \end{aligned}$$

where  $S - (C \mid \neg C \mid eC)$  includes states initiating infinite executions of  $p$ . We are not going to do so, since in constructing correct programs we equally care about the avoidance of abortion and of infinite computations. Therefore we can identify these two cases in our model. Of course, in the denotational model of **Lingua** the abortion was distinguished from infinite looping, because the detection of the latter is not computable.

It may be interesting to see, how on the ground of our relational model, we can express the difference between McCarthy’s and Kleene’s operators of propositional calculus. E.g.

$$\begin{aligned} (A, \neg A) \text{ and-mc } (B, \neg B) &= (A \cap B, \neg A \mid A \cap \neg B) && \text{— McCarthy} \\ (A, \neg A) \text{ and-k } (B, \neg B) &= (A \cap B, \neg A \mid \neg B) && \text{— Kleene} \end{aligned}$$

Now, let  $P$  and  $Q$  represent arbitrary programs and a pair of disjoint sets of states  $(C, \neg C)$  — an arbitrary three-valued partial predicate. Our three structural constructors may be defined as particular cases of the universal set of equations (8.3-1). We recall (Sec. 2.7) that for any set of states  $A$

$$[A] = \{(a, a) \mid a : A\}$$

is a subset of an identity relation (function). Now, the equational definitions of structural constructors are the following:

**Sequential composition** —  $P ; Q$

$$\begin{aligned} X &= P \ Y \\ Y &= Q \end{aligned}$$

Therefore by Theorem 2.4-2:

$$X = P \ Q$$

---

<sup>62</sup> Precisely speaking, any “computable” function. This claim has been known as Church’s thesis. A formal proof of this thesis is shown in [18], and is based on a simple programming language with a (sort of) denotational semantics.

**Conditional composition — if (C, ¬C) then P else Q fi**

$$\begin{aligned} X &= [C] Y \mid [\neg C] Z \\ Y &= P \\ Z &= Q \end{aligned}$$

Therefore by Theorem 2.4-2::

$$X = [C] P \mid [\neg C] Q$$

**Loop — while (C, ¬C) do P od**

$$X = [C] P X \mid [\neg C]$$

As is easy to prove in this case

$$X = ([C] P)^* [\neg C]$$

Summarizing our definitions<sup>63</sup>:

$$\begin{aligned} P ; Q &= P Q \\ \text{if } (C, \neg C) \text{ then } P \text{ else } Q \text{ fi} &= [C] P \mid [\neg C] Q \\ \text{while } (C, \neg C) \text{ do } P \text{ od} &= ([C] P)^* [\neg C] \end{aligned}$$

At the end one methodological remark is necessary. Although in **Lingua** all programs are deterministic, hence correspond to functions rather than relations, in the relational theory of program correctness we shall mainly talk about arbitrary relations (with an exception of **while** loops), since in these cases nondeterminism does not lead to more complicated proof rules.

## 8.4 Procedures and recursion

The next step towards the development of structural-programming techniques was the introduction of procedures and, in particular — recursive procedures. On the ground of the algebra of relations mutually recursive procedures may be regarded as components of a vector of relations  $(R_1, \dots, R_n)$  which is the least solution of a set of fixed-point *polynomial equations* of the form:

$$X_1 = \Psi_1.(X_1, \dots, X_n)$$

...

$$X_n = \Psi_n.(X_1, \dots, X_n)$$

In these equations, each  $\Psi_i(X_1, \dots, X_n)$  is a polynomial, i.e., a combination of variables  $X_1, \dots, X_n$ , with constants, say **A**, **B**, **C**, by composition and union, e.g.,  $A X_1 X_3 B \mid X_5 X_1 C$ . Such sets of equations may be regarded as single fixed-point equations in a CPO of relational vectors ordered component-wise, i.e., in the CPO over the carrier:

$$\text{Rel}(S, S)^{\text{cn}} = \{(R_1, \dots, R_n) \mid R_i : \text{Rel}(S, S)\}$$

Every such set of polynomial equations defines a vectorial function:

$$\begin{aligned} \Psi &: \text{Rel}(S, S)^{\text{cn}} \mapsto \text{Rel}(S, S)^{\text{cn}} \\ \Psi.(R_1, \dots, R_n) &= (\Psi_1.(R_1, \dots, R_n), \dots, \Psi_n.(R_1, \dots, R_n)) \end{aligned}$$

Since each  $\Psi_i$  is continuous in all its variables,  $\Psi$  is continuous as well, and therefore Kleene's theorem applies (Sec. 2.4).

<sup>63</sup> The disjointedness of  $C$  and  $\neg C$  is, of course, not necessary for two last constructors to make sense, but we assume it to assure that these constructors generate functions, if only  $P$  and  $Q$  are functions.

Since the correctness problem for recursive procedures is much more complicated than in the iterative case (see [5]), we shall investigate in Sec. 8.6.2 and Sec. 8.7.2 a simple scheme of a recursive procedure with only one procedural call that corresponds to an equation of the form:

$$X = HXT \mid E \quad (8.4-2)$$

where  $H, T, E : \text{Rel}(S, S)$  are relations called the *head* the *tail* and the *exit* of the procedure, respectively. Although this is certainly not a general scheme for a recursive procedure, it is quite common in practice. This scheme will be referred to as a *simple recursion*.

Notice that (8.4-2) covers the case of the iterative instruction **while-do-od** with  $H = [C]P$ ,  $T = [S]$  and  $E = [\neg C]$ .

## 8.5 Three concepts of program correctness

To express the property of program correctness on the ground of the algebra of binary relations, we shall use two operations of a composition of a relation with a set. Both are similar to sequential compositions of relations defined in Sec. 2.7. In the sequel  $A, B, C, \dots$  will denote subsets of the set of states  $S$  and  $P, Q, R, \dots$  will denote relations in  $\text{Rel}(S, S)$ . Both operations are denoted by the same symbol “ $\bullet$ ”, which has also been used for a composition of functions:

$$\begin{aligned} A \bullet R &= \{s \mid (\exists a:A) a R s\} && \text{— left composition; the image of } A \text{ by } R \\ R \bullet B &= \{s \mid (\exists b:B) s R b\} && \text{— right composition; the coimage of } B \text{ by } R. \end{aligned}$$

In the sequel, the symbol of composition “ $\bullet$ ” will be omitted; hence we shall write  $AR$  and  $RA$ . Intuitively speaking (see Fig. 8.5-1):

- $AR$  is the set of all final states of executions of  $R$  that start in  $A$ ; notice however that some of them may be at the same time final states of executions that start outside  $A$ ,
- $RB$  is the set of all initial states of executions of  $R$  that terminate in  $B$ , but if  $R$  is not a function, then some of them may at the same time generate executions that terminate outside  $B$  or do not terminate at all.

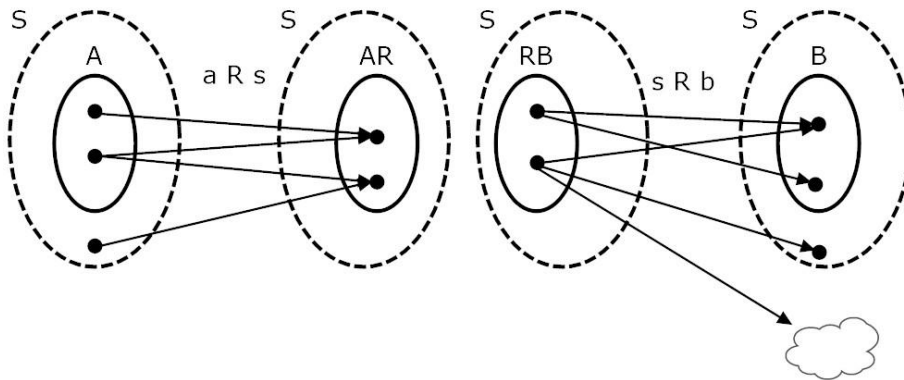


Fig. 8.5-1 Left- and right composition of a set with a relation

Both compositions of a relation with a set have properties similar to that of the composition of two relations. For instance, they are associative:

$$\begin{aligned} A(RQ) &= (AR)Q \\ (RQ)B &= R(QB) \end{aligned}$$

and distributive over unions of sets and relations:

$$\begin{aligned} (A \mid B) R &= (AR) \mid (BR) \\ A (R \mid Q) &= (AR) \mid (AQ) \end{aligned}$$

...

They are also monotone in each argument:

$$\begin{aligned} \text{if } A \subseteq B \text{ then } AR &\subseteq BR \\ \text{if } R \subseteq Q \text{ then } AR &\subseteq AQ \end{aligned}$$

and analogously for right-hand-side composition. In fact, both operations are continuous in each argument. In the sequel, we shall assume that composition binds stronger than union hence we shall write

$$AR \mid BR \text{ instead of } (AR) \mid (BR)$$

**Lemma 8.5-1** *For any  $A, B, C \subseteq S$ , and  $R : \text{Rel}(S, S)$  the following equalities hold:*

1.  $[A]B = A \cap B$
2.  $A[B] = A \cap B$
3.  $(A \cap B)R = A[B]R$
4.  $R(A \cap B) = R[A]B$
5.  $(A \cap B)R \subseteq C$  is equivalent to  $A[B]R \subseteq C$
6. if  $A \subseteq [B]RC$  then  $(A \cap B) \subseteq RC$

Proofs are left to the reader.

Now we are ready to define three fundamental concepts concerning the correctness of programs: *partial correctness*, *weak total correctness*, and *clean total correctness*. All these concepts express the fact that if an input state of a program satisfies certain conditions, then the output state has expected properties. For instance, we may expect that if a list-sorting program is given an appropriate list (precondition), then it will return a sorted list (postcondition).

With every property of states, we can unambiguously associate a set of states satisfying this property. As a consequence, the correctness of a program  $R$  for precondition  $A$  and postcondition  $B$  may be expressed in the algebra of relations and sets in the following way:

$$AR \subseteq B \text{ — } \textit{partial correctness of } R \text{ for precondition } A \text{ and postcondition } B;$$

$$(\forall a:A) (\forall b:B) \text{ if } aRb, \text{ then } b:B$$

$$A \subseteq RB \text{ — } \textit{weak total correctness}^{64} \text{ of } R \text{ for precondition } A \text{ and postcondition } B;$$

$$(\forall a:A) (\exists b) aRb \text{ and } b:B$$

Partial correctness means that every execution that starts in  $A$ , if it terminates, then it terminates in  $B$ . Set  $A$  is called *partial precondition*, and  $B$  is called *partial postcondition*. If  $B$  does not contain error-carrying states then we talk about *clean partial correctness*.

Weak total correctness means that for every state  $a : A$ , there exists an execution that starts in  $a$  and terminates in  $B$ . Set  $A$  is called *weak total precondition*, and  $B$  is called *weak total postcondition*. The adjective “weak” expresses the fact that the existence of an execution from  $a$  to  $B$  does not exclude that other executions starting with  $a$  may terminate outside  $B$  or do not terminate at all. Similarly as in the former case, if  $B$  does not contain error-carrying states then we talk about *weak clean total correctness*.

Both defined concepts of program correctness were historically introduced for deterministic programs, i.e., for the case where  $R$  was a function. In such cases, the inclusion  $A \subseteq RB$  means that each execution of  $R$  that starts in  $A$  terminates in  $B$ . If  $B$  does not include error-carrying states, then this property will be called *clean total correctness* and programs with this property will be said to be *totally correct with clean termination*. Our validating language described in Sec. 9 will include program-construction rules that guarantee clean total correctness of constructed programs.

<sup>64</sup> In the earlier versions of the book the weak total correctness of relations was called just total correctness. Krzysztof Apt convinced us that such wording may lead to misunderstanding. He also pointed out that in [6] written by him with two other authors the notion of weak total correctness is used in a slightly different way. It is used in the context of distributed programs and combines partial correctness with absence of failures and divergence freedom.

As is easy to see, in the non-deterministic case, none of the partial and total correctness is stronger than the other. Indeed, partial correctness does not imply termination, and the existence of one terminating execution from  $a$  to  $B$  does not mean that any terminating execution starting in  $a$  will terminate in  $B$ .

In the deterministic case, however, total correctness obviously implies partial correctness. i.e., for any partial function  $F : S \rightarrow S$ ,

$$A \subseteq FB \text{ implies } AF \subseteq B \quad (8.5-1)$$

The following implication is also true:

$$\text{if } AF \subseteq B \text{ and for every } a : A, F.a \text{ is defined then } A \subseteq FB \quad (8.5-2)$$

Both observations lead to the following theorem:

**Theorem 8.5-1** *If  $F$  is a function, then for any  $A, B \subseteq S$  the following facts are equivalent:*

- $A \subseteq FB$  — *total correctness of  $F$  wrt  $A$  and  $B$*
- $AF \subseteq B$  and  $A \subseteq FS$  — *partial correctness of  $F$  wrt  $A$  and  $B$ , plus termination of  $F$  on  $A$*  ■

*Clean termination* of a deterministic program  $F$  on  $A$  means that  $F$  is a total function on  $A$ , and  $F.a$  never carries an error.

We say that a deterministic program has a *halting property* in  $A$ , if no execution of that program that starts in  $A$  is infinite.

For many “practical programs”, the halting property may be so obvious that it does not need a formal proof. For instance, the program:

```
pre n, m > 0
  x := 1; y := m;
  while x < n
  do;
    x := x+1; y := y*m
  od
post y = m^n
```

obviously halts for every  $n$ . However, there are cases where the halting property may be far from evident, even for very simple programs. One such program is displayed on the front of Warsaw University Library:

```
x := n;
while x > 1
do
  if x mod 2 = 0 then x := x/2 else x := 3x + 1 fi
od
```

Under this program we see the following question: “Why for every  $n > 0$  this program stops?”. This question is, however, not adequate, since today we do not know, if this program has a halting property. It expresses a well-known *Collatz hypothesis* formulated in 1937 and not answered till today. At the date, we are writing these words (February 2024), it was only proved<sup>65</sup> that the hypothesis is true for all  $n < 5 \cdot 2^{66}$ .

<sup>65</sup> One could (naïvely) expect that this result was proved by a simple checking in utilizing an ultra-fast computer. However, as is easy to calculate, if we assume that the execution of Collatz program for any  $n < 5 \cdot 2^{66}$  takes on the average 1 nanosecond, then such a check would take a time longer than  $10^{65}$  times the age of the universe.

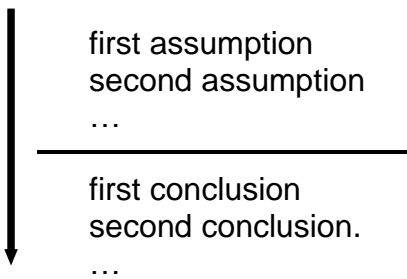
<sup>66</sup> Andrzej Blikle once fell victim to this hypothesis, when he was reporting his work on total correctness of programs at the University of Saarbrücken. When he said that with his method one can easily prove the termination of a program, a listener asked him to illustrate this fact on a simple example, and gave him the Collatz program. Blikle did not know this example, so he wrote the program on the board and proceeded to analyze it. Since he was not able to solve the problem off hand, he said: “I will think about this problem in the evening”. But in the evening he still did not have a proof. What a shame — such a simple program, and he cannot cope with it. After returning to Warsaw he showed the problem to his colleagues, and was enlightened that he was not the only one who was not able to prove the Collatz’s hypotheses.

A similar situation concerns *Fermat's theorem*<sup>67</sup> that was announced in the year 1637 and proved only in 1994 by a British mathematician Andrew Wiles. His proof is 100 pages long and uses an advanced topological theory of elliptic curves. Fermat theorem can be also formulated as a halting problem.

On the ground of the theory of computability, it has been proved by Alan Turing that there is no algorithm which given a program<sup>68</sup> and an input state could check in a finite time, if this program terminates for this input state.

**Theorem 8.5-2** *In the general case, the termination property of programs is not decidable. ■*

In the sequel, proof rules for program correctness will be expressed by showing in which way the correctness of composed programs may be proved by proving the correctness of their components. These rules will be written in the following form:



where the arrow shows the direction of implication. In some rules, we have both-sided arrows, which means that the implication is of the **iff**-type. The list of assumptions and of conclusions are understood as corresponding conjunction.

It should be emphasized in this place that in our approach to program correctness we are not building any “logic of programs” in Hoare’s or Dijkstra’s style. We only construct a set-theoretical model of programs where the latter are represented by binary relations (or functions). On the ground of this model, program correctness is expressed by inclusions of the form  $AP \subseteq B$  or  $AP \subseteq B$ . Then, we formulate and prove some lemmas which may be used either in proving programs correctness, or in building correct programs. In short, these lemmas will be called *proof rules* or *construction rules* depending on the way we shall use them.

In the end, one comment about using single sets of states  $A$  or  $B$ , rather than pairs  $(C, \neg C)$ , to represent three-valued pre- and post-conditions. In fact in using pre- and post-conditions, we are interested only in their “domains of satisfaction”, i.e., in the first elements of each pair  $(C, \neg C)$ . For instance, in proving the correctness of a program with a precondition:

$$1/x > 2 \quad (*)$$

we are only interested in the behavior of the program whenever our precondition is satisfied. We do not care about that behavior in all other cases. If, however, condition  $(*)$  would be used as a boolean expression of an **if-then-else-fi** instruction, then it must be represented by a pair of sets (cf. Rule 8.6.1-2)

## 8.6 Partial correctness

Although our primary concern is total correctness of programs, the methods of proving partial correctness are of interest too since in the deterministic case, proof of total correctness may be reduced to a proof of partial

<sup>67</sup> This theorem claims that for no integer  $n > 2$  there exist three positive integers  $x, y, z$  that satisfy the equality  $x^n + y^n = z^n$ . That theorem had been written in 1637 by Pierre de Fermat on the margin of a book together with a commentary that he found a “marvelously simple proof” of the theorem which was however too long to fit to the margin. The theorem has been proved by Andrew Wiles in 1993, and his proof was more than 100 pages long.

<sup>68</sup> In the original work of A. Turing programs were represented by Turing machines, but since then it became a known fact that for every program there is a (functionally) equivalent Turing machine, and vice versa (e.g. cf. [64]).



correctness plus a proof of termination (cf. (8.5-2)). In turn, although in the general case termination property is not decidable, in many practical cases it may be quite easy to prove.

### 8.6.1 Sequential composition and branching

When defining program correctness proof rules, it is worth distinguishing between two classes of program constructors: *simple constructors* that do not introduce repetition mechanisms and *recursive constructors* that introduce such mechanisms. The former are defined by composition and union of relations; the latter require fixed-point equations. From this perspective, iteration is a particular case of recursion.

The most frequently used simple constructors of programs are sequential composition and branching.

#### Rule 8.6.1-1 Partial correctness of a sequential composition

For arbitrary  $A, B, C, D \subseteq S$  and  $P, Q : \text{Rel}(S, S)$  the following rule is satisfied:

$$\begin{array}{l} \uparrow \\ (1) \ AP \subseteq B \\ (2) \ CQ \subseteq D \\ (3) \ B \subseteq C \\ \hline \downarrow (4) \ A(PQ) \subseteq D \end{array}$$

**Proof** From (1), (2) and the monotonicity of composition

$$(AP)Q \subseteq CQ \subseteq D$$

hence from the associativity of composition

$$A(PQ) \subseteq D.$$

To prove the bottom-to-top implication is sufficient to set

$$B = C = AP$$

Hence  $AP \subseteq B$  and  $BQ = APQ \subseteq D$  ■

#### Rule 8.6.1-2 Partial correctness of if-then-else-fi

For arbitrary  $A, D, C, \neg C \subseteq S$  and  $P, Q : \text{Rel}(S, S)$ , if  $C \cap \neg C = \emptyset$ , then the following rule is satisfied:

$$\begin{array}{l} \uparrow \\ (1) \ (A \cap C)P \subseteq B \\ (2) \ (A \cap \neg C)Q \subseteq B \\ \hline \downarrow (3) \ A \text{ if } (C, \neg C) \text{ then } P \text{ else } Q \text{ fi} \subseteq B \end{array}$$

The proof is obvious.

In the end, three more rules which follow directly from the monotonicity of composition of a set with a relation.

#### Rule 8.6.1-3 Strengthening a partial precondition

For every  $P : \text{Rel}(S, S)$  and any  $A, B, C \subseteq S$  the following rule holds:

$$\begin{array}{l} | \ AP \subseteq B \\ | \ C \subseteq A \\ \hline \downarrow \ CP \subseteq B \end{array}$$

#### Rule 8.6.1-4 Weakening a partial postcondition

For every  $P : \text{Rel}(S, S)$  and any  $A, B, C \subseteq S$  the following rule holds:

$$\begin{array}{l} \text{AP} \subseteq B \\ B \subseteq C \\ \hline \text{AP} \subseteq C \end{array}$$

### Rule 8.6.1-5 The conjunction and disjunction of pre- and postconditions

For every  $P : \text{Rel}(S, S)$  and any  $A, B, C, D \subseteq S$  the following rule holds:

$$\begin{array}{l} \text{AP} \subseteq B \\ \text{CP} \subseteq D \\ \hline (\text{A} \cap \text{C})P \subseteq B \cap D \\ \downarrow (\text{A} \mid \text{C})P \subseteq B \mid D \end{array}$$

In the present section we skip the problem of proving properties of atomic components of programs such as, e.g., assignments or variable declarations since they are not expressible in the model of abstract binary. This issue will be discussed in Sec. 9 where **Lingua-V** enters the game.

## 8.6.2 Recursion and iteration

In order to formulate proof rules for mutually recursive procedures, we generalize the operation of composition of relations with relations and with sets to the case of vectors of respectively relations and sets:

$$(P_1, \dots, P_n) (R_1, \dots, R_n) = (P_1 R_1, \dots, P_n R_n)$$

and analogously for the composition of a relation with sets. In an obvious way, we can also generalize the inclusion of sets to the inclusion of vectors:

$$(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n) \text{ means } A_1 \subseteq B_1 \text{ and } \dots \text{ and } A_n \subseteq B_n$$

For simplicity, the inclusion between vectors of sets is denoted by the same symbol as the inclusion of sets. In the sequel, vectors of sets and relations as well as operations on them will be written with boldface characters.

A vector of relations **R** is said to be *partially correct* wrt the vectors of sets **A** and **B** (with appropriate numbers of elements) iff  $\mathbf{A} \mathbf{R} \subseteq \mathbf{B}$ . The notion of a continuous function is generalized to the case of vectorial functions in an obvious way.

Now we can formulate partial-correctness proof rule in the general case of fixed-points of continuous functions on vectors of relations.

### Rule 8.6.2-1 Partial correctness of a vector of relations defined by a fixed-point equation

For every continuous function  $\Psi : \text{Rel}(S, S)^{cn} \mapsto \text{Rel}(S, S)^{cn}$ , if **R** is the least solution of the equation  $X = \Psi.X$ , then for any **A, B** :  $S^{cn}$  the following rule holds, where  $\emptyset = (\emptyset, \dots, \emptyset)$  is a  $n$ -element vector of empty relations:

$$\begin{array}{l} \uparrow \text{there exists a family of (vectors of) preconditions } \{\mathbf{A}_i \mid i \geq 0\} \\ \text{and a family of (vectors of) postconditions } \{\mathbf{B}_i \mid i \geq 0\} \text{ such that} \\ (1) (\forall i \geq 0) \mathbf{A} \subseteq \mathbf{A}_i \\ (2) (\forall i \geq 0) \mathbf{A}_i \Psi^i. \emptyset \subseteq \mathbf{B}_i \\ (3) \bigcup \{\mathbf{B}_i \mid i \geq 0\} \subseteq \mathbf{B} \\ \hline \downarrow (4) \mathbf{A} \mathbf{R} \subseteq \mathbf{B} \end{array}$$

**Proof** Form Kleene's theorem (Sec. 2.4)

$$\mathbf{R} = \bigcup \{\Psi^i. \emptyset \mid i \geq 0\}$$

Adding the components of (2) sidewise we obtain

$$\bigcup \{\mathbf{A}_i \mid i \geq 0\} \subseteq \bigcup \{\mathbf{B}_i \mid i \geq 0\}$$

hence from (1) and (3), we have (4). To prove the bottom-up implication, we assume

$$B_i = A(\Psi^i.\emptyset) \text{ for } i \geq 0 \text{ and}$$

$$A_i = A$$

■

From this rule, we obtain immediately a rule for single recursion, i.e., where  $n = 1$ :

#### Rule 8.6.2-2 Partial correctness of a relation defined by a fixed-point equation

For every continuous function  $\Psi : \text{Rel}(S,S) \mapsto \text{Rel}(S,S)$ , if  $R$  is the least solution of the equation  $X = \Psi.X$ , then for any  $A, B \subseteq S$  the following rule holds:

$$\begin{array}{l} \uparrow \text{there exists a family of preconditions } \{A_i \mid i \geq 0\} \\ \text{and a family of postconditions } \{B_i \mid i \geq 0\} \text{ such that} \\ (1) (\forall i \geq 0) A_i \Psi^i.\emptyset \subseteq B_i \\ (2) (\forall i \geq 0) A \subseteq A_i \\ (2) \bigcup \{B_i \mid i \geq 0\} \subseteq B \\ \hline (3) AR \subseteq B \\ \downarrow \end{array}$$

We can also formulate more specific rules for each particular polynomial function, e.g., for the simple-recursion constructor as defined in Sec. 8.4. Below two versions of such a rule:

#### Rule 8.6.2-3 Partial correctness of a relation defined by simple recursion (version 1)

For any  $H, T, E : \text{Rel}(S,S)$ , if the relation  $R$  is the least solution of the equation

$$X = HXT \mid E$$

then for any  $A, B \subseteq S$  the following rule holds:

$$\begin{array}{l} \uparrow \text{there exists a family of preconditions } \{A_i \mid i \geq 0\} \\ \text{and a family of postconditions } \{B_i \mid i \geq 0\} \text{ such that} \\ (1) (\forall i \geq 0) A_i H_i E T_i \subseteq B_i \\ (2) (\forall i \geq 0) A \subseteq A_i \\ (2) \bigcup \{B_i \mid i \geq 0\} \subseteq B \\ \hline (3) AR \subseteq B \\ \downarrow \end{array}$$

The proof follows immediately from Rule 8.6.2-2 and from the fact that, as is easy to prove,

$$R = \bigcup \{H^i E T^i \mid i \geq 0\} \quad \blacksquare$$

The following top-down-implication rule with a stronger assumption may be useful as well:

#### Rule 8.6.2-4 Partial correctness of a relation defined by simple recursion (version 2)

For any  $H, T, E : \text{Rel}(S,S)$ , if the relation  $R$  is the least solution of the equation

$$X = HXT \mid E$$

then for any  $A, B \subseteq S$  the following rule holds:

$$\begin{array}{l} (1) (\forall Q) (AQ \subseteq B \text{ implies } A(HQT) \subseteq B) \\ (2) AE \subseteq B \\ \hline (3) AR \subseteq B \\ \downarrow \end{array}$$

**Proof** From (1) and (2) we can prove by induction that for every  $i \geq 0$ :

$$A(H^i E T^i) \subseteq B$$

and, therefore, by side-wise summation, we get (3). ■

### Rule 8.6.2-5 A Partial correctness of a relation defined by simple recursion (version 3)

For any  $H, T, E : \text{Rel}(S, S)$ , if the relation  $R$  is the least solution of the equation

$$X = HXT \mid E$$

then for any  $A, B \subseteq S$  the following rule holds:

$$\left| \begin{array}{l} (1) AH \subseteq A \\ (2) AE \subseteq B \\ (3) BT \subseteq B \\ \hline (4) AR \subseteq B \end{array} \right|$$

**Proof** The three inclusions (1), (2), and (3) imply that for any  $i > 0$ , we have

$$A(H^i E T^i) \subseteq A E T^i \subseteq B T^i \subseteq B. \blacksquare$$

Now, let us recall that by

**while** ( $C, \neg C$ ) **do**  $P$  **od**

the least solution of the equation

$$X = [C]PX \mid [\neg C].$$

Setting  $H = [C]P$ ,  $T = [S]$  and  $E = [\neg C]$  from both general rules we can draw rules for while-do-od iteration:

### Rule 8.6.2-6 Partial correctness of while-do-od loop (version 1)

For every relation  $P : \text{Rel}(S, S)$ , any disjoint  $C, \neg C \subseteq S$ , and any  $A, B \subseteq S$  the following rule holds:

$$\left| \begin{array}{l} \text{there exists a family of postconditions } \{B_i \mid i \geq 0\} \text{ such} \\ \text{that} \\ (1) (\forall i \geq 0) A ([C]P)^i [\neg C] \subseteq B_i \\ (2) \bigcup \{B_i \mid i \geq 0\} \subseteq B \\ \hline (3) A \text{ while } (C, \neg C) \text{ do } P \text{ od} \subseteq B \end{array} \right|$$

### Rule 8.6.2-7 Partial correctness of while-do-od loop (version 2)

For every relation  $P : \text{Rel}(S, S)$ , any disjoint  $C, \neg C \subseteq S$ , and any  $A, B \subseteq S$  the following rule holds:

$$\left| \begin{array}{l} (1) (\forall Q) (AQ \subseteq B \text{ implies } A [C]QP \subseteq B) \\ (2) A[\neg C] \subseteq B \\ \hline (3) A \text{ while } (C, \neg C) \text{ do } P \text{ od} \subseteq B \end{array} \right|$$

■

In the literature, the following rule is also well known, although it is usually formulated for the case of two-valued predicates, i.e. where  $C \mid \neg C = S$

**Rule 8.6.2-8 Partial correctness of while-do-od loop (version 3)**

For every relation  $P : \text{Rel}(S, S)$ , for any disjoint  $C, \neg C \subseteq S$ , any  $A, B \subseteq S$ , the following rule is satisfied:

$$\begin{array}{l}
 \uparrow \text{there exists } N \subseteq S \text{ (called loop invariant) such that:} \\
 (1) (N \cap C) P \subseteq N \\
 (2) A \subseteq N \\
 (3) N [\neg C] \subseteq B \\
 \hline
 (4) A \text{ while } (C, \neg C) \text{ do } P \text{ od } \subseteq B \\
 \downarrow
 \end{array}$$

■

**Proof** Let (1) – (3) be satisfied. Since

$$(N \cap C) P = N [C] P$$

from (1) we can prove by induction:

$$N([C]P)^i \subseteq N \text{ for all } i \geq 0$$

Therefore and from (2)

$$A([C]P)^i \subseteq N \text{ for all } i \geq 0$$

hence from (3)

$$A([C]P)^i [\neg C] \subseteq N[\neg C] \subseteq B \text{ for all } i \geq 0$$

In summing these inclusions sidewise, we get (4). Now assume that (4) is satisfied and let us set:

$$(5) N = A([C]P)^*$$

Therefore and from (4) we get  $N[\neg C] \subseteq B$ , hence (3). In turn (5) is equivalent to

$$N = A \mid A([C]P)^+,$$

hence (2). To prove (1) notice that:

$$(N \cap C)P = N[C]P = A[C]P \mid A([C]P)^+[C]P = A([C]P)^+ \subseteq N \quad \blacksquare$$

**8.7 Weak total correctness**

Rules for weak total correctness are used to prove that if an input state of a program satisfies a precondition, then at least one execution of that program will terminate with postconditions satisfied. If a program is deterministic, then weak total correctness coincides with clean total correctness which means that the unique execution of a program terminates with a state satisfying a postcondition.

**8.7.1 Sequential composition and branching****Rule 8.7.1-1 Weak total correctness of a composition**

For any  $A, B, C, D \subseteq S$  and  $P, Q : \text{Rel}(S, S)$  the following rule holds:

$$\begin{array}{l}
 \uparrow (1) A \subseteq PB \\
 (2) C \subseteq QD \\
 (3) B \subseteq C \\
 \hline
 (4) A \subseteq (PQ)D \\
 \downarrow
 \end{array}$$

**Proof.** From (1), (2) and (3) we immediately have:

$$A \subseteq PB \subseteq PC \subseteq P(QD) = (PQ) D.$$

Now assume that  $A \subseteq (PQ)D$ , which means that  $A \subseteq P(QD)$ . Assuming  $B = C = QD$  we get (1) and (2). ■

**Rule 8.7.1-2 Weak total correctness of if-then-else**

For any  $A, B, C, \neg C \subseteq S$  and  $P, Q : \text{Rel}(S, S)$ , if  $C \cap \neg C = \emptyset$ , then the following rule is satisfied:

$$\begin{array}{l} \uparrow (1) A \cap C \subseteq PB \\ (2) A \cap \neg C \subseteq QB \\ (3) A \subseteq C \mid \neg C \\ \hline \downarrow (4) A \subseteq \text{if } (C, \neg C) \text{ then } P \text{ else } Q \text{ fi } B \end{array}$$

**Proof.** Let (1) – (3) be satisfied. Then:

$$\begin{array}{l} [C] (A \cap C) \subseteq [C] PB \\ [\neg C] (A \cap \neg C) \subseteq [\neg C] QB \end{array}$$

Adding the inclusions sidewise:

$$[C] (A \cap C) \mid [\neg C] (A \cap \neg C) \subseteq [C] PB \mid [\neg C] QB = ([C]P \mid [\neg C] Q) B$$

The following equalities are also true

$$[C] (A \cap C) = A \cap C$$

and analogously for  $\neg C$ . Hence and from (3)

$$[C] (A \cap C) \mid [\neg C] (A \cap \neg C) = (A \cap C) \mid (A \cap \neg C) = A$$

and finally

$$(4) A \subseteq [C] PB \mid [\neg C] QB$$

In turn, (4) implies  $A \subseteq C \mid \neg C$ , and from (4) and the fact that  $C$  and  $\neg C$  are disjoint, follow (1) and (2). ■

Observe the assumption (3) in our rule. In the case of classical predicates where  $C \mid \neg C = S$ , this condition is a tautology.

In the end, three more rules for pre- and postconditions analogous to the respective rules for partial correctness.

**Rule 8.7.1-3 The strengthening of a weak total precondition**

For every  $P : \text{Rel}(S, S)$  and any  $A, B, C \subseteq S$  the following rule holds:

$$\begin{array}{l} A \subseteq PB \\ C \subseteq A \\ \hline \downarrow C \subseteq PB \end{array}$$

**Rule 8.7.1-4 The weakening of a weak total postcondition**

For every  $P : \text{Rel}(S, S)$  and any  $A, B, C \subseteq S$  the following rule holds:

$$\begin{array}{l} A \subseteq PB \\ B \subseteq C \\ \hline \downarrow A \subseteq PC \end{array}$$

**Rule 8.7.1-5 The conjunction and disjunction of conditions**

For every  $F : S \rightarrow S$  and any  $A, B, C, D \subseteq S$  the following rule holds:

$$\begin{array}{l} A \subseteq FB \\ C \subseteq FD \\ \hline \downarrow \begin{array}{l} A \cap C \subseteq F(B \cap D) \\ A \mid C \subseteq F(B \mid D) \end{array} \end{array}$$



The proofs of the last three rules follow directly from the definitions of total correctness. Our last rule in this section concerns resilient conditions.

### Rule 8.7.1-6 Propagation of resilient conditions

For every  $P : \text{Rel}(S, S)$  and any  $A, B, C \subseteq S$  the following rule holds:

$$\begin{array}{l} \downarrow \begin{array}{l} (1) A \subseteq PB \\ (2) CP \subseteq C \end{array} \\ \hline A \cap C \subseteq P(B \cap C) \end{array}$$

In this rule,  $C$  is said to be *resilient to*  $P$ , because its satisfaction is not violated by  $P$ . This rule, although quite simple, has a practical value, since it will be applied in all situations where a certain property of a state once established, remains in force till the end of the execution of a program. E.g., once we declare a variable it remains declared during the whole (remaining) lifetime of the hosting program. The proof of this rule is the following:

From (1) by Rule 8.7.1-3 we have  $A \cap C \subseteq PB$ . Consequently, if  $a : A \cap C$  then there exists a state  $b$  such that  $a P b$  and  $b : B$ . At the same time, since  $a : C$ , then by (2)  $b : C$ , hence  $b : B \cap C$ . ■

## 8.7.2 Recursion and iteration

Similarly, as in the case of partial correctness, we start from the case of a general recursive operator.

### Rule 8.7.2-1 Weak total correctness of a vector defined by a general fixed-point equation

For every continuous function  $\Psi : \text{Rel}(S, S)^{\text{cn}} \mapsto \text{Rel}(S, S)^{\text{cn}}$ , if  $R$  is the least solution of  $X = \Psi.X$ , then the following rule holds, where  $\emptyset = (\emptyset, \dots, \emptyset)$ :

$$\begin{array}{l} \uparrow \begin{array}{l} \text{there exists a family of preconditions } \{A_i \mid i \geq 0\} \\ \text{and a family of postconditions } \{B_i \mid i \geq 0\} \text{ such that} \\ (1) (\forall i \geq 0) A_i \subseteq (\Psi^i.\emptyset)B_i \\ (2) A \subseteq \bigcup \{A_i \mid i \geq 0\} \\ (3) (\forall i \geq 0) B_i \subseteq B \end{array} \\ \hline (4) A \subseteq RB \end{array}$$

**Proof** If  $R$  is the least fixed point of  $\Psi$ , then from the continuity of  $\Psi$

$$(4) R = \bigcup \{\Psi^i.\emptyset \mid i \geq 0\}$$

Adding sidewise inclusions (1) we have

$$\bigcup \{A_i \mid i \geq 0\} \subseteq \bigcup \{(\Psi^i.\emptyset) B_i\}$$

Hence from (2) and (3), we have (4). Now assume that  $A \subseteq RB$  which means that

$$A \subseteq \bigcup \{\Psi^i.\emptyset \mid i \geq 0\} B$$

Let for  $i \geq 0$

$$A_i = (\Psi^i.\emptyset) B \text{ and}$$

$$B_i = B$$

Then obviously (1), (2), and (3) are satisfied. ■

From this rule for  $n = 1$ , we immediately conclude the next rule

**Rule 8.7.2-2 Weak total correctness of a relation defined by a general fixed-point equation**

For every continuous function  $\Psi : \text{Rel}(S, S) \mapsto \text{Rel}(S, S)$ , if  $R$  is the least solution of an equation  $X = \Psi.X$ , then the following rule holds:

$$\begin{array}{l}
 \uparrow \text{there exists a family of preconditions } \{A_i \mid i \geq 0\} \\
 \text{and a family of postconditions } \{B_i \mid i \geq 0\} \text{ such that} \\
 (1) (\forall i \geq 0) A_i \subseteq (\Psi^i.\emptyset)B_i \\
 (2) A \subseteq \bigcup \{A_i \mid i \geq 0\} \\
 (3) (\forall i \geq 0) B_i \subseteq B \\
 \hline
 (4) A \subseteq RB
 \end{array}$$

**Rule 8.7.2-3 Weak total correctness of a relation defined by simple recursion (version 1)**

If relation  $R$  is the least solution of the equation  $X = H X T \mid E$  then the following rule holds:

$$\begin{array}{l}
 \uparrow \text{there exists a family of preconditions } \{A_i \mid i \geq 0\} \\
 \text{and a family of postconditions } \{B_i \mid i \geq 0\} \text{ such that} \\
 (1) (\forall i \geq 0) A_i \subseteq (H^i E T^i) B_i \\
 (2) A \subseteq \bigcup \{A_i \mid i \geq 0\} \\
 (3) (\forall i \geq 0) B_i \subseteq B \\
 \hline
 (4) A \subseteq RB
 \end{array}$$

**Proof** Define

$$\Psi.X = H X T \mid E$$

In this case

$$\Psi^0.\emptyset = E$$

$$\Psi^1.\emptyset = \Psi.(\Psi^0.\emptyset) = H (\Psi^0.\emptyset) T \mid E = H E T \mid E$$

$$\Psi^2.\emptyset = \Psi.(\Psi^1.\emptyset) = H (\Psi^1.\emptyset) T \mid E = H (\Psi^1.\emptyset) T \mid E = H^2 E T^2 \mid H^1 E T^1 \mid E$$

Therefore, by induction, for any  $n \geq 0$

$$\Psi^i.\emptyset = \bigcup \{ H^i E T^i \mid i=1,2,\dots,n \} \mid E = \bigcup \{ H^i E T^i \mid i=0,1,\dots,n \}$$

Now, by (1) and the monotonicity of composition of a relation with a set, we have for every  $i \geq 0$

$$A_i \subseteq H^i E T^i B_i \subseteq (\bigcup \{ H^i E T^i \mid i=0,\dots,n \}) B_i \subseteq (\Psi^i.\emptyset) B_i$$

From this inclusion together with (2), (3) and Rule 0-2, we conclude

$$A \subseteq RB$$

In turn, if the inclusion is satisfied, then we set

$$A_i = (\Psi^i.\emptyset) B$$

$$B_i = B$$

With this settings (1) and (3) are obviously satisfied, and (2) is satisfied because

$$A \subseteq RB \subseteq \bigcup \{ \Psi^i.\emptyset \mid i \geq 0 \} B = \bigcup \{ (\Psi^i.\emptyset) B \mid i \geq 0 \} = \bigcup \{ A_i \mid i \geq 0 \} \blacksquare$$

**Rule 8.7.2-4 Clean total correctness of a function defined by simple recursion (version 2)**

If  $F$  is the least solution of the equation  $X = HXT \mid E$  where  $H$ ,  $T$ , and  $E$  are functions and the domains of  $H$  and  $E$  are disjoint, then the following rule holds:

$$\begin{array}{l} \downarrow \begin{array}{l} (1) (\forall Q) (AQ \subseteq B \text{ implies } A(HQT) \subseteq B) \\ (2) AE \subseteq B \\ (3) A \subseteq FS \end{array} \\ \hline (4) A \subseteq FB \end{array}$$

**Proof** As is easy to prove, for any  $H$ ,  $T$ , and  $E$  the least solution of our equation is

$$U\{H^n E T^n \mid n \geq 0\}$$

and if additionally  $H$ ,  $T$ , and  $E$  are functions and the domains of  $H$  and  $E$  are disjoint, then this solution is a function. Now, by (1), (2) and the Rule 8.6.2-4,  $AF \subseteq B$ , i.e.,  $F$  is partially correct wrt  $A$  and  $B$ . Since (3) means that  $F$  is total on  $A$ , by Theorem 8.5-1 we can claim that it is totally correct wrt  $A$  and  $B$ . ■

From Rule 8.7.2-3 we can immediately derive our first rule about **while-do-od** instruction based on the observation that **while** ( $C$ ,  $\neg C$ ) **do**  $P$  **od** is the least solution of the equation

$$X = [C]PX \mid [\neg C].$$

Let then  $R$  be the least solution of this equation, i.e.,

$$R = ([C]P)^*[\neg C].$$

**Rule 8.7.2-5 Clean total correctness for nondeterministic while-do-od**

$$\begin{array}{l} \uparrow \begin{array}{l} \text{there exists a family of preconditions } \{A_i \mid i \geq 0\} \\ \text{and a family of postconditions } \{B_i \mid i \geq 0\} \text{ such that} \\ (1) (\forall i \geq 0) A_i \subseteq ([C]P)^i[\neg C] B_i \\ (2) A \subseteq U \{A_i \mid i \geq 0\} \\ (3) (\forall i \geq 0) B_i \subseteq B \end{array} \\ \hline (4) A \subseteq RB \end{array}$$

The most commonly known version of a rule for **while-do-od** concerns a deterministic case, and does not require the construction of two infinite families of conditions. It is also based on a well-known method of proving the halting property of a loop. First, we introduce two auxiliary concepts.

We say that a function  $F : S \rightarrow S$  has *limited replicability property* in a set  $N \subseteq S$ , if there exists no infinite sequence of the form:  $s, F.s, F.(F.s), \dots$  in  $N$ .

A partially ordered set  $(U, >)$  is said to be *well-founded*, if there is no infinite decreasing sequence in it, i.e., a sequence  $u_1 < u_2 < \dots$ . The following obvious lemma is useful in proving the limited replicability of a function  $F : S \rightarrow S$ .

**Lemma 8.7.2-1** If there exists a well-founded set  $(U, <)$  and a function  $K : N \mapsto U$  such that for any  $a : N$ ,  $F.a = !$ ,  $F.a : N$  and

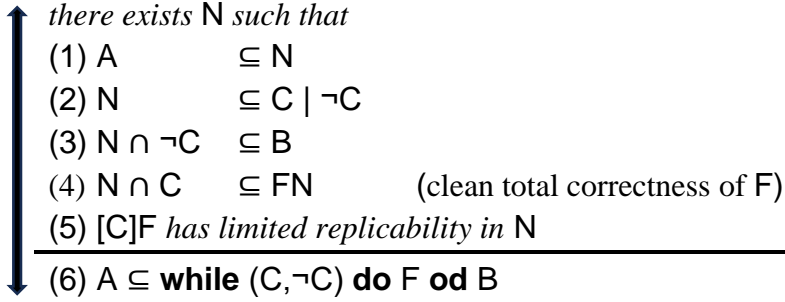
$$K.a > K.(F.b)$$

then  $F$  has limited replicability in  $N$ . ■

Now we can formulate our rule.

**Rule 8.7.2-6 Clean total correctness of a deterministic while-do-od loop**

For any function  $F : S \rightarrow S$ , any  $A, B \subseteq S$ , and any disjoint  $C, \neg C \subseteq S$



**Proof** We prove the conclusion (6), which is

$$A \subseteq ([C]F)^*[\neg C]B.$$

Consider any  $s_0:A$ , which by (1) implies  $s_0:N$ . Since  $[C]F$  is a function, by (5)  $([C]F)^k.s_0$  is undefined for some  $k>0$ . Let  $n \geq 0$  be the least such that  $([C]F)^{n+1}.s_0$  is undefined. Let  $s_i = ([C]F)^i.s_0$ ,  $i = 0, 1, \dots, n$ . Using (4), it follows by easy induction on  $i$  that  $s_i : N$ . Hence  $s_n : C \mid \neg C$  by (2). If  $s_n : C$  then by (4)  $([C]F)^{n+1}.s_0$  would be defined, so  $s_n : \neg C$ , and by (3)  $s_n : B$  – which completes the proof.

To prove the converse implication set  $N = A([C]F)^*$ . ■

In the end of this section let us reemphasize that rules 8.7.2-3 and 8.7.2-5 have been formulated (and are satisfied) for arbitrary relations, hence are valid not only in deterministic, but also in nondeterministic cases. At the same time, rules 8.7.2-4 and 8.7.2-6 are true only in deterministic cases.

## 9 VALIDATING PROGRAMMING

### 9.1 A discipline of validating programming

This section's title paraphrases the title of the famous Edsger Dijkstra book *A Discipline Of Programming* ([51]), which became a breakthrough in thinking about program writing in 1976 (see quotation in Sec. 8.1). In Sec. 9, we summarize our early thoughts about this discipline seen from a denotational perspective.

Generally speaking, by *validating programming*, we shall mean such program construction technique that ensure clean total-correctness of programs relative to their specifications. In our approach, programs and their specifications will constitute syntactic components of *metaprograms*. This technique was already announced in Sec. 8.1 and its set-theoretical foundations were described in sections from 8.2 to 8.7. An example of a metaprogram (in its future syntax) may be the following:

|  |   |         |
|--|---|---------|
| <pre>pre x,k is integer and-k k &gt; 0:   x := 0;   asr x = 0 rsa;   while x+1 ≤ k do x := x+1 od post x = k</pre> | <p>— precondition (specification)</p> <p>— assertion (specification)</p> <p>— postcondition (specification)</p> | (9.1-1) |
|--|---|---------|

A metaprogram is said to be correct if its program is cleanly, totally correct relative to its specification. This definition will be formalized in Sec. 9.3.3.

Very roughly speaking, validating programming consists in developing correct metaprograms from other correct metaprograms or from some patterns by means of sound construction rules. The soundness of such rules means that if we use them to transform a correct metaprogram or to put together some correct metaprograms, then the resulting metaprogram is correct (Sec. 9.4).

From an abstract point of view the process of validating programming in **Lingua** may be regarded as proving theorems of a formalized *theory of denotations of Lingua*. In this book we shall call it **D-theory**.

In a general case, the language of a formalized theory includes two basic syntactic categories: *terms* and *formulas*. Both may be *free* or *ground*. Examples of terms and formulas in formalized arithmetic may be the following where  $x$  and  $y$  are variables:

|   |  |
|---|--|
| <pre>x + y 2 + 3 x &lt; 1+y 2 &lt; 3 + 75</pre> | <p>— a free term,</p> <p>— a ground term,</p> <p>— a free formula,</p> <p>— a ground formula</p> |
|---|--|

By a *valuation of variables* we mean a function that assigns values, in this case integers, to variables. In arithmetic, terms represent functions from valuations to integers, and formulas, functions from valuations to classical truth values **tt** and **ff**. The set of integers constitutes the *universe* of arithmetic<sup>69</sup>. A formula that is satisfied for all valuations is said to be *valid*. Examples of valid formulas in arithmetic may be:

2 < 3    or  
x < x + 1.

---

<sup>69</sup> Precisely it is the universe of a *model* of our theory, but since we shall not refer to the theory of models, we shall call it, for simplicity, just a universe of the theory.

Let's now try to figure out what the language of our **D-theory** looks like? Since it is a theory of the denotations of **Lingua**, its terms should evaluate to denotations and its formulas should describe properties of denotations. Consequently, the universe of **D-theory** is not a single set but a family of sets including all carriers of the algebra of denotations listed in Sec. 6.1. As we see, **D-theory** is many-sorted. A little later we shall add some further carriers to its universe.

Note now that from this perspective metaprograms like (9.1-1) constitute ground formulas of our theory. They are simply true or false since they are statements about concrete programs. In turn, an example of a free formula of **D-theory** may be

```
pre con1 :
  ins
post con2
```

(9.1-2)

where `con1`, `con2` and `ins` — called *D-variables* — are running over the denotations of conditions or instructions respectively. The denotations of instructions are well-known, and the denotations of conditions will be partial functions from states to the elements of  $\{(\text{tt}, \text{'boolean'}), (\text{ff}, \text{'boolean'})\} \mid \text{Error}$ .

From this perspective metaprograms will belong to the category of formulas in **D-theory**. Since they are 2-valued, the logic used in **D-theory** will be classical. Formulas in **D-theory** will be otherwise called *metaconditions*.

Terms of **D-theory** they will include all syntactic (colloquial) categories of **Lingua** (see Sec. 9.2.6) plus an additional category of *conditions* (Sec. 9.2). The latter express properties of states. Syntactically they include boolean expressions of **Lingua** where McCarthy's connectives are replaced by Kleene's connectives<sup>70</sup>, plus some other “boolean-like” expressions that are not available at the level of programs.

The language of **D-theory** will be denoted by **Lingua-V**; V for “validating”.

Similar to the case of **Lingua**, we shall not attempt to define any “complete” **Lingua-V**. We only sketch its syntax and semantics, mainly by showing examples and defining their meanings more or less intuitively. We shall regard this language half-formally as a mathematical notations, rather than a formalized language with a grammar and a function of semantics. Of course, a formalization of **Lingua-V** will be necessary when it comes to building an ecosystem for programmers in **Lingua-V** equipped with an automated theorem prover.

In the sequel, the variables of our language, 2-valued logical connectives, the special operator  $\Rightarrow$  and parentheses that do not belong to the level of **Lingua**, will be printed in Arial black (possibly in bold), whereas all other elements of **Lingua-V** in *Ariel Narrow green*. An example of a formula with free variables in **Lingua-V** may be the following:

**((pre prc : spr post poc) and (poc  $\Rightarrow$  poc-1)) implies (pre prc : spr post poc-1)**

In the end of this section two methodological remarks:

The first remark concerns the fact that although in **Lingua-V** we use 3-valued boolean expressions and conditions, in **D-theory** we use 2-valued classical logic<sup>71</sup>. Our conditions only “happen to look like” logical formulas, but in **Lingua-V** they are just terms. Our formulas, i.e., metaconditions, are two-valued, and describe properties of (the denotations of) conditions and programs.

The second remark concerns the way in which we shall “read” (understand) our free terms and formulas. Formally, D-variables included in them run over denotations, hence such terms and formulas represent functions from valuations to denotations or to truth values. At the same time, however, we may regard them as *schemes* representing families of terms or formulas of a common shape. E.g. if we talk about conditional instructions of the form

<sup>70</sup> Due to the use of Kleene's connectives we gain the commutativity of alternative and conjunction. We lose the error transparency wrt to their right-hand-side arguments, but this fact is not harmful, since conditions are not evaluated (cf. Sec. 2.10).

<sup>71</sup> Readers interested in 3-valued logics developed on the ground of McCarthy's calculus of predicates are referred to [65] where such logics are discussed.

**if** exp **then** ins-1 **else** ins-2 **fi**

we may regard this “piece of metasyntax” either as a term in **D-theory** with free variables **exp**, **ins-1** and **ins-2**, or as a *scheme of an instruction*, where **exp** stands for arbitrary value-expression and **ins-i**’s stand for arbitrary instructions. In the sequel we shall frequently use such schemes without calling them explicitly in this way, unless the context does not indicate what we mean.

## 9.2 Conditions

### 9.2.1 General assumptions about conditions

Denotationally *conditions* represent partial functions from states to boolean values or errors:

$\text{cod} : \text{ConDen} = \text{WfState} \rightarrow \text{BooValE}$  the denotations of conditions

For future use we introduce the following notations:

$\text{val} : \text{BooValE} = \{\text{tv}, \text{fv}\} \mid \text{Error}$

$\text{tv} = (\text{tt}, \text{'boolean'})$

$\text{fv} = (\text{ff}, \text{'boolean'})$

By

$\text{con} : \text{Condition} = \dots$

we shall denote the (colloquial) syntactic domain of conditions. As D-variables running over **Condition** we shall also use:

- **prc** to denote *preconditions*,
- **poc** to denote *postconditions*.

The syntactic domain of conditions of a “practical” **Lingua-V** may be very large and strongly dependent on the domain of applications of such a language. Therefore, we restrict our further discussion about conditions to only these categories of conditions that seem to be universal for validated programming, i.e., domain independent.

Our first assumption is that the domain of conditions includes all boolean expressions of **Lingua** but with McCarthy’s logical connective replaced by Kleene’s connectives (see Sec. 2.10). In this way we gain the commutativity of conjunction and disjunction. Note that, since conditions will never be evaluated, but only “proved”, we do not need to care about the necessity of a parallel evaluation of the arguments of conjunction and alternative.

Consequently we assume that the domain of conditions is closed under 3-valued logical connectives in Kleene’s version, i.e.:

$(\text{con1} \text{ and-k } \text{con2}), (\text{con1} \text{ or-k } \text{con2}), (\text{not-k } \text{con})$

belong to **Condition** for any **con1**, **con2**, **con** : **Conditions**.

The semantics of conditions will be denoted by square brackets [ ], and besides we also introduce the concept of a *truth domain* of a condition:

$[\text{con}] : \text{WfState} \rightarrow \text{BooValE}$

the denotation of **con**

$\{\text{con}\} = \{\text{sta} \mid [\text{con}].\text{sta} = \text{tv}\}$

the *truth domain* of **con**

From now on we shall use square brackets to denote the semantics of all our terms and formulas, assuming that a context will always indicate which semantics we mean.

In the sequel we shall distinguish between four semantic categories (properties) of conditions. A condition **con** will be said to be:

- *error transparent* — if  $[\text{con}].\text{sta} = \text{error.sta}$  whenever **sta** carries an error,
- *error negative* — if  $[\text{con}].\text{sta} = \text{fv}$  whenever **sta** carries an error,
- *error positive* — if  $[\text{con}].\text{sta} = \text{tv}$  whenever **sta** carries an error,



- *error sensitive* — if it is error transparent or error negative

The majority of conditions that we shall define in the sequel will be error-sensitive, however, not error-sensitive conditions will also play a certain role in our approach (e.g. right-hand algorithmic conditions in Sec. 9.2.7). Error-positive conditions may be useful if we want to claim that a metaprogram generates errors, e.g., if we are building error traps (Sec. 6.5.3).

In the end, we assume that **Condition** includes a special condition **NT** — spelled out as “nearly true” — that is true for all states that do not carry errors, and otherwise it returns the error carried by the state, i.e.,

```
[NT].sta =
  is-error.sta  → error.sta
  true          → tv
```

Note that this condition is error-sensitive.

## 9.2.2 Value-oriented conditions

*Value-oriented conditions* describe the properties of values assigned to variables and attributes in states. We assume that syntactically they include all value-expressions with boolean values but with Kleene’s operators, such as, e.g.,

$x+1 < 2*z$  **and-k**  $z > 0$ ,

However, some value-oriented conditions may have no counterparts in boolean expressions. Typical examples are equality conditions of the form

$vex-1 = vex-2$ ,

where  $vex-1$  and  $vex-2$  are value expressions with not-comparable values (see Sec. 6.4.1).

Note that at the level of boolean expressions, we usually do not allow comparison of structural values, such as e.g., lists, arrays, objects or databases, since this might be computationally too expensive. However, we allow such comparisons at the level of conditions, since in this case we do not check (compute) the equalities, but only claim them (prove them) to express properties of programs. Another example of a condition that is not a boolean expressions may be

**increasingly ordered real** (*ide*)

This condition is satisfied if *ide* points to a list of real numbers ordered increasingly

## 9.2.3 Cov-oriented conditions

The mechanism of type-covering relations imposes the necessity of checking types’ compatibilities in four following situations:

1. when a value is assigned to a reference by an assignment instruction,
2. when a value of an actual parameter is assigned to the reference of a formal parameter by the mechanism of entering a procedure call,
3. when a reference of an actual parameter is assigned to the corresponding formal parameter by the mechanism of entering a procedure call,
4. when a reference of a formal parameter is passed (back) to the corresponding actual parameter.

In cases 1., 2. and 3 we refer to the current covering relation, but in case 4., i.e., at the exit of a procedure, we have to refer to the covering relation of a call-time state (Sec. 6.6.3.5). We recall that due to the assumption about the structure of our programs (Sec. 6.3), the call-time covering relation equals the declaration-time relation.

To illustrate this case assume that at the exit of an imperative-procedure call we have a local terminal state (see Sec. 6.6.3)

$lt-sta = ((lt-cle, lt-pre, lt-cov), lt-sto)$



**true**  $\rightarrow$  tv

To define the denotation of the second one we need an auxiliary function

**list-of-ide** : ForPar  $\mapsto$  LisOfIde

that given a (list of) formal parameter, i.e., a syntactic element, e.g.,:

**x, y, z as real, n, m, p as integer**

returns the list of identifiers

**x, y, z, n, m, p**

(cf., a similar construction in Sec. 6.6.3.3). We skip a formal definition of this function. Now:

```
[fpa well-valued in coe].sta =
  is-error.sta  $\rightarrow$  error.sta
  let
    cov = [coe].sta
    (ide-1,...,ide-n) = list-of-ide.fpa
  cov : Error  $\rightarrow$  cov
  let
    (env, (obn, dep, ota, sft, 'OK')) = sta
    obn.ide-i = ?  $\rightarrow$  'variable not declared' for i = 1;n
    dep.(obn.ide-i) = ?  $\rightarrow$  'variable not initialized' for i = 1;n
    ( $\forall i$ )(obn.ide-i VRA.cov dep.(obn.ide-i))  $\rightarrow$  tv
  true  $\rightarrow$  fv
```

Note that the acceptance of values by corresponding references is checked wrt the type-covering relation indicated by **coe** which needs not coincide with the current relation carried by **sta**.

Given these new categories of conditions we can express the fact that at the exit of a procedure body formal reference parameters are well-valued wrt a call-time covering relation:

```
prc-call  $\Rightarrow$  coe is current and
poc-body  $\Rightarrow$  fpa-r well-valued in coe
```

Here we use a metapredicate  $\Rightarrow$  to be read as ‘is stronger than’ that we shall formalize in Sec. 9.3. This technique will be used in Sec. 9.4.6.3 where we formulate a rule for the creation of procedure declarations that lead to correct procedure calls.

Our last special condition in this section concerns the enrichment of a covering relation by a new pair of types. We recall (cf. Sec. 5.4.2 and 6.7.5) that two types may be added to a covering relation if:

1. they are different,
2. they do not belong to this relation,
3. they are either both datatypes or both object types,
4. if they are object types, i.e., identifiers, then they are names of declared classes.

The following condition checks if a given pair of types can be added to a current covering relation:

```
[ consistent(tex-1, tex-2) ].sta =
  is-error.sta  $\rightarrow$  error.sta
  [tex-i].sta : Error  $\rightarrow$  [tex-i].sta for i = 1,2
  let
    typ-i = [tex-i].sta for i = 1,2
    ((cle, pre, cov), sto) = sta
  typ-1 : DatTyp and typ-2 : ObjTyp  $\rightarrow$  'types not comparable'
  typ-2 : DatTyp and typ-1 : ObjTyp  $\rightarrow$  'types not comparable'
  (typ-1, typ-2) : cov  $\rightarrow$  'types already in covering relation'
  typ-1, typ-2 : DatTyp  $\rightarrow$  tv
  cle.typ-i = ?  $\rightarrow$  'object types must be declared' for i = 1,2
```

**true** **→ tv**

Note that all (schemes of) conditions defined above are error-transparent.

## 9.2.4 Basic value-, type- and reference-oriented conditions

Conditions of this category, except the last one, describe properties of output states of non-procedural categories of declarations. The last condition does not belong to this group, but is listed here because it is a necessary prerequisite for all declarations to execute cleanly. Below we show some typical examples of conditions associated with values, types and references, but excluding these of Sec. 9.2.3:

- |   |   |
|---|---|
| (1) <b>att</b> at-ide <b>is</b> tex <b>in</b> cl-ide <b>as</b> pst, | at-ide is declared with tex in class cl-ide as pst  |
| (2) ty-ide <b>is type in</b> cl-ide,                                | ty-ide is declared as type constant in class cl-ide                                       |
| (3) <b>var</b> ide <b>is</b> tex,                                   | ide is a declared variable of type tex  |
| (4) rex <b>is reference</b> ,                                       | reference expression rex evaluates cleanly  |
| (5) vex <b>is value</b> ,   | value expression vex evaluates cleanly  |
| (6) vex <b>is</b> tex   | vex evaluates cleanly and its value is of type indicated by tex (which evaluates cleanly) |
| (7) tex <b>is type</b> ,  | type expression tex evaluates cleanly   |
| (8) cli <b>is class</b> ,   | cli is either empty-class or an identifier of a declared class                            |
| (9) ide <b>child of</b> cli,  | ide is an identifier of a declared class which is a child of class indicated by cli       |
| (10) <b>consistent</b> (tex1, tex2)                                 | tex1 and tex2 evaluate cleanly and corr. types are not in <b>TTA.cov</b> relation         |
| (11) ide <b>is free</b>   | ide has not been declared   |

We assume that all these conditions are error transparent. We define only two of these categories of conditions since the remaining ones seem fairly obvious.

```
[ att at-ide is tex in cl-ide as pst ].sta =
  is-error.sta           → error.sta
  [tex].sta : Error       → [tex].sta
  let
    ((cle, pre, cov), sto) = sta
    typ                     = [tex].sta
  cle.cl-ide = ?           → 'class unknown'
  let
    (cl-ide, tye, mee, obn) = cle.cl-ide
    obn.at-ide = ?         → 'attribute unknown'
  let
    (tok, (at-typ, at-ori) = obn.at-ide
    typ ≠ at-typ           → 'types not compatible'
    pst = 'private' and ar-ori ≠ cl-ide → 'privacy status not adequate'
    pst = 'public' and ar-ori ≠ $      → 'privacy status not adequate'
  true                     → tv
```

Proceeding to the definition of (9) we recall that a child class named **ch-ide** is a child of a parent class named **pa-ide**, if it inherits all procedure signatures and all attributes of the parent class.

```
[ch-ide child of pa-ide].sta =
  is-error.sta           → error.sta
  let
    ((cle, pre, cov), sto) = sta
    cle.pa-ide = ?        → 'parent class unknown'
    cle.ch-ide = ?        → 'child class unknown'
  let
    pa-cla = cle.pa-ide
    (ch-ide, [ ], fu-mee, ch-obn) = make-funding-class.ch-ide. pa-cla (see Sec. 6.7.3)
    (ch-ide, ch-tye, ch-mee, ch-obn) = cle.ch-ide
    fu-mee ⊆ ch-mee       → 'signatures not compatible'
```

dom.pa-obn  $\not\subseteq$  dom.ch-obn  $\rightarrow$  'attributes not compatible'  
 true  $\rightarrow$  tt

At the end a comment about condition (5). For it to be satisfied, all variables and attributes in **vex** must be declared and initialized. It is also worth noticing that the satisfaction of (5) implies that the evaluation of **vex** won't generate an error message, e.g., an overflow.

## 9.2.5 Procedure-oriented conditions

Conditions of this category describe the effects of procedure declarations. Here are typical examples:

- (1) pr-ide (val fpv ref fpr) begin body end imperative in cl-ide,
- (2) fu-ide (val fpv ref tex) begin body return vex end functional in cl-ide,
- (3) ob-ide (val fpv ref ob-ide) begin body end objectional in cl-ide,
- (4) procedure cl-ide.pr-ide opened
- (5) (pass actual val apa-v ref apa-r to formal val fpa-v ref fpa-r with cl-ide) @ con

The denotation of (1) is the following:

```
[pr-ide (val fpc-v ref fpc-r) begin body end imperative in cl-ide].sta =
  is-error.sta  $\rightarrow$  error.sta
  let
    ((cle, pre, cov), sto) = sta
    cle.cl-ide = ?  $\rightarrow$  'class unknown'
  let
    (cl-ide, tye, mee, obn) = cle.cl-ide
    mee.pr-ide = ?  $\rightarrow$  'pre-procedure unknown'
  let
    declared-pre-proc = mee.pr-ide
    expected-pre-proc = create-imp-pre-pro.([fpd-v], [fpd-r], [body])
    declared-pre-proc  $\neq$  expected-pre-proc  $\rightarrow$  fv
  true  $\rightarrow$  tv
```

Our condition claims three facts:

1. cl-ide is a name of a declared class,
2. pr-ide is a name of a pre-procedure in this class,
3. pre-procedure pointed by pr-ide is equal to a pre-procedure that would be created by a declaration  
**proc** pr-ide (val fpc-v, ref fpc-r) begin body end.

Note that in 3. we do not claim that the body of the declared pre-procedure is body, but that the declared procedure (a denotational element) is identical with a procedure generated by **proc** pr-ide (val fpc-v, ref fpc-r) begin body end. That is, therefore, not a claim about syntax, but about its “denotational effect”. This condition is also not computable. The definitions for cases (2) and (3) are analogous.

It is to be noted in this place that conditions (1) – (3) all syntactically include bodies of corresponding pre-procedures, also formally they do not imply that the pre-procedures in question have been created “from these bodies”. At the same time, however, each of these conditions may appear in an assertion or a postcondition of a program in only two cases:

1. if it appears in the precondition,
2. if a corresponding pre-procedure has been declared in the program (cf. Sec. 9.4.4).

It practically means that condition (1) – (3) will (in practical situations) always include the bodies of actually declared procedures. These conditions belong to the category of *inherent conditions* (cf. Sec. 9.3.5), which once satisfied remain satisfied till the end of program execution.

Condition (4) claims that pre-procedure `pr-ide` declared in class `cl-ide` gave rise — due to the opening declaration — to a procedure assigned to procedure indicator (`cl-ide`, `pr-ide`) in the procedure environment of the current state. We skip an obvious definition.

The last category of conditions has an algorithmic character (see Sec. 9.2.7), and will be used to describe the effect of an action of passing actual parameters to formal parameters in a procedure call. In its definition we shall refer to function `pass-actual` defined in Sec. 6.6.3.4.

```
[ pass actual val apa-v ref apa-r to formal val fpa-v ref fpa-r with cl-ide ) @ con ].sta =
  is-error.sta      → error.sta
  let
    (env, sto) = sta
    new-sto    = pass-actual.(fpa-v, fpa-r, apa-v, apa-r, cl-ide).env.sto
    is-error.new-sto → error.new-sto
  let
    new-sta = (env, new-sto)
  true      → [con].new-sta
```

We recall that `cl-ide` is a name of a class. It is necessary here to mark `new-sto` with `cl-ide` as a local store of a procedure call.

A state that satisfies condition (5) guarantees that starting with it, the execution of `pass-actual` will terminate cleanly, and the output state will satisfy condition `con`. We shall use this condition in Sec. 9.4.6.3, where we formulate a rule for constructing correct procedure calls.

## 9.2.6 Assertions and specified programs

As we have already seen in Sec. 8, the rules of the development of correct metaprograms are lemmas which guarantee the correctness of some metaprograms provided that their components were correct. In an abstract case where programs are represented by binary relations between (abstract) states, correctness of programs was expressed by pre- and postconditions. However, at the level of a programming language we may also need to talk about properties of their intermediate states. To do that we introduce of a syntactic category of *assertions* with the following domain:

`asr` : Assertion = **asr** Condition **rsa**

The semantic of assertions is the following:

```
[asr] : WfState → WfState
[asr con rsa].sta =
  is-error.sta      → sta
  [con].sta = ?      → ?
  [con].sta : Error  → sta ◀ [con].sta
  [con].sta = fv     → sta ◀ 'assertion not satisfied'
  true             → sta
```

Note that an error message will be generated by assertions in two situations:

1. when the value of the condition is an error,
2. when the condition is not satisfied.

An assertion may be regarded as a filter that is transparent for error-carrying states and for states satisfying the included condition, and otherwise either aborts the execution of a program or loops.

Another new concept that we shall need in the future will be used in building rules for the development of class declarations (Sec. 6.7.3). By an *anchored class transformer* we mean an imperative element with the following syntactic domain:

`act` : AncClaTra = ClaTra **in** Identifier

and the following semantics:



$[\text{ctr in ide}] : \text{WfState} \rightarrow \text{WfState}$   
 $[\text{ctr in ide}] = [\text{ctr}].\text{ide}.$

The identifier in this structure is called an *anchor*. We recall that the denotations of class transformers constitute the following domain:

$\text{ctd} : \text{ClaTraDen} = \text{Identifier} \mapsto \text{WfState} \rightarrow \text{WfState}.$

which means that a transformer, when given a (class) identifier, becomes a state-to-state function, i.e., a denotation of an anchored class transformer.

Now, we are ready to define *specified programs* and their specified components. Intuitively they are imperative components of programs with nested assertions. Assertion may be “inserted” into a program everywhere “in the position” of a declaration or an instruction.

Below we formally define the corresponding syntactic domains. We also introduce the concepts of *on-zones* and *of-zones* of specinstructions with semantics defined a little later.

|  |  |   |
|--|--|---|
| $\text{sin} : \text{SpeIns} =$                                   |  | <i>specified instructions or specinstructions</i>       |
| Instruction  |  |   |
| Assertion  |  |   |
| SpeIns ; SpeIns  |  |   |
| <b>asr</b> con in SpeIns <b>rsa</b>                              |  | <i>on-zones declarations</i>                            |
| <b>off</b> con in SpeIns <b>on</b>                               |  | <i>off-zones declarations</i>                           |
| <b>if</b> ValExp <b>then</b> SpeIns <b>else</b> SpeIns <b>fi</b> |  |   |
| <b>if-error</b> ValExp <b>then</b> SpeIns <b>fi</b>              |  |   |
| <b>while</b> ValExp <b>do</b> SpeIns <b>od</b>                   |  |   |
| <b>skip-ins</b>  |  |   |
| $\text{sde} : \text{SpeDec} =$                                   |  | <i>specified declarations or specdeclarations</i>       |
| Declaration  |  |   |
| Assertion  |  |   |
| SpeDec ; SpeDec  |  |   |
| <b>skip-dec</b>  |  |   |
| $\text{sct} : \text{SpeClaTra} =$                                |  | <i>specified class transformers or spectransformers</i> |
| AncClaTra  |  |   |
| Assertion  |  |   |
| SpeClaTra ; SpeClaTra  |  |   |
| <b>skip-sct</b>  |  |   |
| $\text{spp} : \text{SpeProPre} =$                                |  | <i>specified program preambles or specpreambles</i>     |
| SpeDec   |  |   |
| SpeIns   |  |   |
| SpeProPre ; SpeProPre  |  |   |
| <b>skip-spp</b>  |  |   |
| $\text{spr} : \text{SpePro} =$                                   |  | <i>specified programs or specprograms</i>               |
| SpeProPre ; <b>open procedures</b> ; SpeIns                      |  |   |
| SpeProPre  |  |   |
| SpeClaTra  |  |   |

Sometimes we shall need to express the fact that an assertion **asr** con **rsa** is to be satisfied not only in one particular location of a specinstruction sin, but during the “whole execution” of sin, i.e., by all its intermediate states with the exclusion of local states of procedure calls. In such a case, instead of “physically” inserting an assertion into sin in all expected locations, we shall use an *on-zone instruction* of the form

**asr** con in sin **rsa**



where **con** will be called a *zone assertion*. Typical situations where we want to insure the satisfaction of an assertion in a zone take place in data-base programming, where zone assertions are known as *integrity constraints*. In the same area of applications we sometime wish to “switch off” a zone assertion, to perform an operation that temporarily “spoils” an integrity constraint. In such a case we shall write

**off con in sin on**

to indicate the range of the off-zone. Examples of the use of both concepts in metaprogram derivation are shown in Sec. 9.5.1 and 9.5.2.

We shall not formalize the ranges of assertions, since this would lead to too many technicalities. We only wish to signalize the idea, in order to use it in simple situations.

In the end it is worth noticing that the semantics of zones is not compositional, since in a general case the denotation

[ **asr con in sin rsa** ]

can’t be described as a function of [sin] and [con]. As an example consider two following specinstructions:

**asr x > 0: x := x rsa**                      and  
**asr x > 0: x := -x ; x := -x rsa**

The denotations of their included instructions are identical, but the denotations of declarations are not.

In the end a comment about the denotations of specprograms and of their spec-components. These denotations are, of course, the same as of programs and their components, i.e., partial state-to-state functions. Their semantics is implicit in the fact that assertions are just instructions and therefore may appear in programs wherever other instructions may. The on- and off-zones may be regarded at the moment as colloquialisms for compound instructions where every semicolon outside pre-procedure bodies has been replaced by a corresponding assertion followed by a semicolon, e.g.:

**asr x > 0 in**  
  y := x+1 ;  
  z := z + 2 ;  
  **call** employee.promote ( **val** y, z **ref** x )  
**rsa**

stands for

y := x+1 ;  
**asr x > 0 rsa ;**  
  z := z + 2 ;  
**asr x > 0 rsa ;**  
  **call** employee.promote ( **val** y, z **ref** x ).

Analogously, in an off-zone we removes the earlier inserted assertions.

## 9.2.7 Algorithmic conditions

*Algorithmic conditions* pay a special role in our approach, since they allow for uniform definitions of partial and total correctness of programs (see Sec. 9.3.3). We define their syntax in the following way:

|   |  |
|---|--|
| <b>con</b> : AlgCondition =<br>SpePro @ Condition  <br>Condition @ SpePro | (9.2.7-1)<br><i>left-algorithmic conditions</i> <sup>72</sup><br><i>right-algorithmic conditions</i> |
|---|--|

and their semantics as follows:

---

<sup>72</sup> Left algorithmic conditions, although not called in this way, constituted a fundament of *algorithmic logic* developed at Warsaw University in the decades 1970. and 1980. (see [10]).

$$[ ] : \text{AlgCondition} \mapsto \text{WfState} \mapsto \{\text{tv}, \text{fv}\}$$

$$\begin{aligned} [\text{spr} @ \text{con}].\text{sta} = \\ (\exists \text{sta1} : \{\text{con}\}) [\text{spr}].\text{sta} = \text{sta1} &\rightarrow \text{tv} \\ \text{true} &\rightarrow \text{fv} \end{aligned} \quad \text{in short: } [\text{spr}].\text{sta} : \{\text{con}\}$$

This condition is satisfied, if  $[\text{spr}].\text{sta} = !$  and  $[\text{spr}].\text{sta}$  satisfies  $\text{con}$ . If  $\text{con}$  is error-sensitive, then the satisfaction of  $\text{spr} @ \text{con}$  means that  $\text{spr}$  terminates cleanly.

$$\begin{aligned} [\text{con} @ \text{spr}].\text{sta} = \\ (\exists \text{sta1} : \{\text{con}\}) [\text{spr}].\text{sta1} = \text{sta} &\rightarrow \text{tv} \\ \text{true} &\rightarrow \text{fv} \end{aligned}$$

This condition is satisfied by all output states of  $\text{spr}$  if “their” input states satisfy  $\text{con}$ .

As we see, none of our algorithmic conditions is error-transparent. However:

- if  $\text{spr}$  is error transparent and  $\text{con}$  is error sensitive, then  $\text{spr} @ \text{con}$  is error negative,
- $\text{con} @ \text{spr}$  may be not error sensitive, even in  $\text{con}$  and  $\text{spr}$  are both error-transparent since the outputted  $\text{sta}$  may carry an error despite that “its” input state does not.

Since algorithmic conditions are two-valued by their definitions<sup>73</sup> they are unambiguously identified by their truth domains, which are the following:

$\{\text{spr} @ \text{con}\}$  is the set of all input states that cause  $\text{spr}$  to terminate with output states that satisfy  $\text{con}$ ,  
 $\{\text{con} @ \text{spr}\}$  is the set of all output states of  $\text{spr}$  for input states that satisfy  $\text{con}$ .

Left algorithmic condition may be not computable, even for computable  $\text{spr}$  and  $\text{con}$ , since the termination property of programs is not decidable. The following obvious equalities also hold (for the definition of ‘ $\bullet$ ’ see Sec. 2.7):

$$\begin{aligned} \{\text{spr} @ \text{con}\} &= [\text{spr}] \bullet \{\text{con}\} \\ \{\text{con} @ \text{spr}\} &= \{\text{con}\} \bullet [\text{spr}] \end{aligned}$$

We assume that the domain **Condition** is closed under both operations of building algorithmic conditions, which in particular means, that conditions in algorithmic conditions may be algorithmic themselves. To “justify” the use of  $@$  in condition (5) of Sec. 9.2.5, we shall include this condition in the category of algorithmic conditions, although it does not include a specified program.

## 9.3 Metaconditions

### 9.3.1 A taxonomy of metaconditions

The art of programming in **Lingua** is not reduced to writing programs. Equally important is writing and using conditions since the relationships between programs and conditions describe properties of programs, and in particular, their correctness. Such relationships are formalized by *metaconditions* already mentioned in Sec. 9.1. They belong to the level of **Lingua-V** that is the language of **D-theory**. We recall that specified programs (with their components) and conditions constitute the category of terms of this theory and metaconditions are formulas. Consequently (sound) program-construction rules will be theorems in **D-theory**.

As in usual logic, metaconditions may be *atomic* or *compound*. The latter are built from the former by means of classical logical connectives and quantifiers. We shall define five following groups of atomic metaconditions:

1. *relational metaconditions* that represent binary relations between conditions,
2. *metaprograms* that describe properties of specprograms,

<sup>73</sup> We could have made them three-valued, but we do not need to do so, since in using algorithmic conditions we shall refer to their truth domains only.

3. *behavioral metaconditions* that describe properties of conditions relative to specinstructions or specprograms,
4. *temporal metaconditions* that describe properties of conditions related to their execution-time in correct metaprograms,
5. *language-dependent metaconditions* that describe properties of conditions which are not related to programs, but depend on a programming language where they are used.

### 9.3.2 Relational metaconditions

Relational metaconditions are built by means of the following four relations between conditions which we call *relational metapredicates*:

|   |   |   |
|---|---|---|
| $\text{con1} \Rightarrow \text{con2}$     | iff (def) $\{\text{con1}\} \subseteq \{\text{con2}\}$ | <i>metaimplication</i> ; stronger than        |
| $\text{con1} \Leftrightarrow \text{con2}$ | iff (def) $\{\text{con1}\} = \{\text{con2}\}$         | <i>weak equivalence</i>                       |
| $\text{con1} \sqsubseteq \text{con2}$     | iff (def) $[\text{con1}] \subseteq [\text{con2}]$     | <i>better definedness</i> ; more defined than |
| $\text{con1} \equiv \text{con2}$          | iff (def) $[\text{con1}] = [\text{con2}]$             | <i>strong equivalence</i>                     |

To better understand their nature let's analyze the following examples, where we assume that the evaluation of square root  $\sqrt[2]{x}$  generates an error if  $x < 0$  is true, and  $x$  is a real-number variable:

$$\begin{aligned}
 x > 0 \text{ and-} k \sqrt[2]{x} > 2 &\equiv x > 4 \\
 \sqrt[2]{x} > 2 &\Leftrightarrow x > 4 \quad \text{but } \equiv \text{ does not hold,} \\
 \sqrt[2]{x} > 4 &\Rightarrow x > 3 \quad \text{but neither } \Leftrightarrow \text{ nor } \sqsubseteq \text{ holds.}
 \end{aligned}$$

Note that in the first line, if the value of  $x$  is not a real number, then both sides generate the same error message: 'real value expected',

If we assume that for negative  $x$ , function  $\sqrt[2]{x}$  is undefined rather than generates an error, then the following relation holds:

$$\sqrt[2]{x} < 2 \sqsubseteq x < 4 \quad \text{but neither } \equiv \text{ nor } \Leftrightarrow \text{ holds,}$$

The following rather obvious relationships hold between our metapredicates:

|   |         |   |
|---|---------|---|
| $\text{con1} \equiv \text{con2}$          | iff     | $\text{con1} \sqsubseteq \text{con2}$ and $\text{con2} \sqsubseteq \text{con1}$ |
| $\text{con1} \Leftrightarrow \text{con2}$ | iff     | $\text{con1} \Rightarrow \text{con2}$ and $\text{con2} \Rightarrow \text{con1}$ |
| $\text{con1} \sqsubseteq \text{con2}$     | implies | $\text{con1} \Leftrightarrow \text{con2}$                                       |
| $\text{con1} \equiv \text{con2}$          | implies | $\text{con1} \sqsubseteq \text{con2}$   |
| $\text{con1} \Leftrightarrow \text{con2}$ | implies | $\text{con1} \Rightarrow \text{con2}$   |

It is important to understand the difference between three implication-like constructors that belong to two different linguistic (and logical) levels:

|                     |   |  |
|---------------------|---|--|
| 1. <b>implies-k</b> | : Condition x Condition $\mapsto$ Condition | constructor of conditions in <b>Lingua-V</b> |
| 2. $\Rightarrow$    | : Condition x Condition $\mapsto$ {tt, ff}  | metaimplication in <b>MetaLingua</b>         |
| 3. <b>implies</b>   | : {tt, ff} x {tt, ff} $\mapsto$ {tt, ff}    | classical implication in <b>MetaLingua</b>   |

The relationship between these three constructors is the following, provided that  $\text{con1}$  is error sensitive:

$$(\text{con1} \text{ implies-k } \text{con2}) \equiv \text{NT} \text{ implies } \text{con1} \Rightarrow \text{con2}.$$

Indeed, let  $\text{sta} : \{\text{con1}\}$  which means that  $[\text{con1}].\text{sta} = \text{tv}$  and  $\text{sta}$  does not carry an error. This implies that  $[(\text{con1} \text{ implies-k } \text{con2})].\text{sta} = \text{tv}$  (since  $\text{sta}$  carries no error), and therefore then  $[\text{con2}].\text{sta} = \text{tv}$  must hold.

If, however,  $\text{con1}$  is not error-sensitive, then  $\text{sta}$  may satisfy  $\text{con1}$  and carry an error, in which case  $(\text{con1} \text{ implies-k } \text{con2}).\text{sta} : \text{Error}$ , and, therefore  $[\text{con2}].\text{sta}$  need not be  $\text{tv}$ .

On the other hand, even if  $\text{con1}$  is error-sensitive:

$$\text{con1} \Rightarrow \text{con2} \text{ does not need to imply } (\text{con1} \text{ implies-k } \text{con2}) \equiv \text{NT}.$$

Indeed, despite that the metaimplication  $\sqrt[2]{x} > 4 \Rightarrow x > 3$  holds, the condition

$$\sqrt[2]{x} > 4 \text{ implies-}k \ x > 3$$

is undefined for  $x < 0$ .

Besides the two-argument metapredicates, we also define three-argument metapredicates which will be used in the development of correct metaprograms<sup>74</sup>:

$$\begin{aligned} \text{con1} \equiv \text{con2} \quad \text{whenever con} \quad &\text{means} \quad \text{con} \text{ and-}k \text{ con1} \equiv \text{con} \text{ and-}k \text{ con2} \\ \text{con1} \Leftrightarrow \text{con2} \quad \text{whenever con} \quad &\text{means} \quad \text{con} \text{ and-}k \text{ con1} \Leftrightarrow \text{con} \text{ and-}k \text{ con2} \\ \text{con1} \Rightarrow \text{con2} \quad \text{whenever con} \quad &\text{means} \quad \text{con} \text{ and-}k \text{ con1} \Rightarrow \text{con} \text{ and-}k \text{ con2} \end{aligned}$$

In all these cases, we say that **con** constitutes a *logical context* or simply a *context* for the metapredicate that it precedes. We shall also say that the *equivalence*  $\text{con1} \equiv \text{con2}$  is *satisfied under the condition con* and analogously for a weak equivalence and metaimplication. E.g. the following metapredicates are satisfied:

$$\begin{aligned} n > x^2 &\equiv \sqrt[2]{n} > x \text{ whenever } (n \geq 0 \text{ and-}k x \geq 0) \\ n > x^2 &\Leftrightarrow \sqrt[2]{n} > x \text{ whenever } x \geq 0 \end{aligned}$$

The context is usually a condition in whose range we want to replace one condition by another one.

Conditions constitute an algebra with Kleene's logical connectives as constructors. In turn, relational metapredicates constitute relations between conditions. Below we describe major relationships between them.

**Lemma 9.3.2-1** *Relations  $\equiv$  and  $\Leftrightarrow$  are both equivalences in the set of conditions, i.e., they are reflexive, symmetric, and transitive.*

**Lemma 9.3.2-2** *Strong equivalence is a congruence wrt **and-k**, **or-k** and **not-k**, i.e., the replacement of a subcondition of a condition by a strongly equivalent one result a condition strongly equivalent to the initial one.*

**Lemma 9.3.2-3** *Weak equivalence is a congruence wrt **and-k** and **or-k**.*

Weak equivalence is not a congruence wrt negation which means that

$$\text{con1} \Leftrightarrow \text{con2} \quad \text{does not imply} \quad \text{not-}k \text{ con1} \Leftrightarrow \text{not-}k \text{ con2}$$

For instance, although

$$\sqrt[2]{x} > 2 \Leftrightarrow x > 4$$

is satisfied, the metacondition

$$\sqrt[2]{x} \leq 2 \Leftrightarrow x \leq 4$$

is not, since for  $x = -1$  the right-hand-side condition evaluates to **tv**, but on the left-hand side, we have an error.

**Lemma 9.3.2-4** *The operators **and-k** and **or-k** are strongly and weakly commutative, i.e.*

$$\begin{aligned} \text{con1} \text{ and-}k \text{ con2} &\equiv \text{con2} \text{ and-}k \text{ con1} \\ \text{con1} \text{ or-}k \text{ con2} &\equiv \text{con2} \text{ or-}k \text{ con1} \\ \text{con1} \text{ and-}k \text{ con2} &\Leftrightarrow \text{con2} \text{ and-}k \text{ con1} \\ \text{con1} \text{ or-}k \text{ con2} &\Leftrightarrow \text{con2} \text{ or-}k \text{ con1} \end{aligned}$$

Note that since strong equivalence implies weak equivalence we could have written only the two first equations.

**Lemma 9.3.2-5** *The operators **and-k** and **or-k** are strongly and weakly associative, i.e.*

$$\begin{aligned} (\text{con1} \text{ and-}k \text{ con2}) \text{ and-}k \text{ con3} &\equiv \text{con1} \text{ and-}k (\text{con2} \text{ and-}k \text{ con3}) \\ (\text{con1} \text{ or-}k \text{ con2}) \text{ or-}k \text{ con3} &\equiv \text{con1} \text{ or-}k (\text{con2} \text{ or-}k \text{ con3}) \\ (\text{con1} \text{ and-}k \text{ con2}) \text{ and-}k \text{ con3} &\Leftrightarrow \text{con1} \text{ and-}k (\text{con2} \text{ and-}k \text{ con3}) \\ (\text{con1} \text{ or-}k \text{ con2}) \text{ or-}k \text{ con3} &\Leftrightarrow \text{con1} \text{ or-}k (\text{con2} \text{ or-}k \text{ con3}) \end{aligned}$$

<sup>74</sup> Of course, with each such metapredicate, we can associate in an obvious way a corresponding constructor of (the syntax of) metaconditions.

**Lemma 9.3.2-6** *The operator **and-k** is strongly and weakly left-hand-side and right-hand-side distributive wrt to **or-k** and vice versa, i.e..*

$$\begin{aligned} \text{con1 } \mathbf{and-k} (\text{con2 } \mathbf{or-k} \text{ con3}) &\equiv (\text{con1 } \mathbf{and-k} \text{ con2}) \mathbf{or-k} (\text{con1 } \mathbf{and-k} \text{ con3}) \\ \text{con1 } \mathbf{or-k} (\text{con2 } \mathbf{and-k} \text{ con3}) &\equiv (\text{con1 } \mathbf{or-k} \text{ con2}) \mathbf{and-k} (\text{con1 } \mathbf{or-k} \text{ con3}) \\ (\text{con2 } \mathbf{or-k} \text{ con3}) \mathbf{and-k} \text{ con1} &\equiv (\text{con2 } \mathbf{and-k} \text{ con1}) \mathbf{or-k} (\text{con3 } \mathbf{and-k} \text{ con1}) \end{aligned}$$

...

**Lemma 9.3.2-7** *The de Morgan's laws for **and-k** and **or-k** and for the negation of quantifiers are satisfied with strong equivalence and weak equivalence.*

The classical laws that are not satisfied in the Kleene's calculus are the law of excluded middle and the law of noncontradiction. Instead we have their weaker versions.

**Lemma 9.3.2-8** For every state  $\text{sta}$ :

$$\begin{aligned} \text{if } [\text{con}].\text{sta} = ! \quad \text{then} \quad & [\text{con } \mathbf{or-k} (\mathbf{not} \text{ con})].\text{sta} \neq \text{fv} & \text{i.e. } \{\text{con } \mathbf{or-k} (\mathbf{not} \text{ con})\} = \{\} \\ \text{if } [\text{con}].\text{sta} = ! \quad \text{then} \quad & [\text{con } \mathbf{and-k} (\mathbf{not} \text{ con})].\text{sta} \neq \text{tv} \end{aligned}$$

The ideas presented in this section were published by A. Blikle in the decade 1980 in [28] and [31], and their development towards three-valued deductive theories was investigated in [65] by B. Konikowska, A. Tarlecki and A.Blikle

### 9.3.3 Metaprograms and their correctness

Using metaimplication and algorithmic conditions, we can easily express partial and total correctness of a specprogram  $\text{spr}^{75}$  for a precondition  $\text{prc}$  and a postcondition  $\text{poc}$ :

$$\begin{aligned} \text{prc } @ \text{ spr} &\Rightarrow \text{poc} & \text{partial correctness} \\ \text{prc} &\Rightarrow \text{spr } @ \text{ poc} & \text{total correctness} \end{aligned}$$

As we see,

- $\text{prc } @ \text{ spr}$  is the *strongest partial postcondition* for  $\text{spr}$  and  $\text{prc}$ ,
- $\text{spr } @ \text{ poc}$  is the *weakest total precondition* for  $\text{spr}$  and  $\text{poc}^{76}$ .

Note also that if  $\text{poc}$  is error-sensitive, then total correctness insures non-abortion. In such a case it is called *clean total correctness*. Having said that we may assume that metaprograms

$$\mathbf{pre} \text{ prc} : \text{spr } \mathbf{post} \text{ poc}$$

stand for  $\text{prc} \Rightarrow \text{spr } @ \text{ poc}$ . In other words:

$$\mathbf{pre} \text{ prc} : \text{spr } \mathbf{post} \text{ poc} \text{ iff (def) } \text{prc} \Rightarrow \text{spr } @ \text{ poc}$$

In an analogous way we define the categories of *metainstructions*, and *metadeclarations*.

Although metaprograms are formally true or false, we shall use a more intuitive wording saying that they are *correct* or *incorrect*.

The last concept that we define in this section concerns the reactions to errors by metaprograms. A metaprogram is said to be *error-sensitive* if all conditions that appear in this program, i.e., precondition, postcondition and assertions, are error-sensitive (Sec. 9.2.1). Error-sensitive metaprograms play an important role in our approach, because the majority of programs that we shall consider have this property.

<sup>75</sup> Of course, specprograms are deterministic, i.e., represent functions.

<sup>76</sup> These concepts are due to Edsger W. Dijkstra (see [50] and [51]).

### 9.3.4 Behavioral metaconditions

The following behavioral metaconditions are important for the development of correct metaprograms, The corresponding constructors will be called *behavioral metapredicates*:

|                               |     |  |  |
|-------------------------------|-----|--|--|
| con <b>insures LR of</b> ins  | iff | [sin] has limited replicability (LR) in {con} (see Sec. 8.7.2) |  |
| con <b>resilient to</b> spr   | iff | con @ spr $\Rightarrow$ con                                    | — con is <i>resilient to</i> spr, if its satisfaction is not violated by spr,                                      |
| con <b>consumed by</b> spr    | iff | con $\Rightarrow$ spr @ <b>not-k</b> con                       | — con is <i>consumed by</i> spr, if it is like a raw material that assures execution but disappears after it,      |
| con <b>catalyzing for</b> spr | iff | con $\Rightarrow$ spr @ con                                    | — con is <i>catalyzing for</i> spr, if it is like a chemical catalyzer — it assures execution but is not consumed, |
| con <b>essential for</b> spr  | iff | con $\equiv$ spr @ <b>NT</b>                                   | — con is <i>essential for</i> spr if it is the weakest preconditions that ensures a clean termination of spr.      |

Since specprograms are by definition deterministic (represent functions), catalyzing conditions are resilient, but not vice versa. To illustrate the defined metapredicates consider a simple metaprogram:

```

pre (x is free) and-k (var y is integer) :
let x be real tel;
x := 17,3
...
post (x is real) and-k (var y is integer) and-k (x = 17,3)

```

(9.3.4-2)

The following relations hold for the specprogram of this metaprogram:

- **var y is integer** is resilient but not catalyzing,
- **x is free** is consumed by the declaration of **x** and is essential for it.
- **var x is real** is catalyzing and essential for **x := 17,3**.

Note that the catalyzing condition for the assignment is not essential, since is not the weakest. The essential condition for **x := 17,3** is **var x is real**.

It is to be emphasized that although we considered a metaprogram in our example, all illustrated properties concern relations between a condition and the specprogram of this metaprogram, and do not depend on the fact that our exemplary metaprogram is correct.

### 9.3.5 Temporal metaconditions

Our next group of metaconditions concerns the satisfaction of conditions during the executions of correct metaprograms, i.e., depending on where and when they are evaluated. If we consider a sequence of consecutive states generated during an execution of a program, and if we evaluate a condition in each of these states one after another, the truth value of such condition may vary depending on the execution time. To analyze such time-dependent properties of conditions without talking about sequences of states<sup>77</sup>, we introduce an auxiliary concept of a *cut* of a specprogram.

<sup>77</sup> A semantics where program denotations are sets of sequences of states was introduced and investigated by Andrzej Blikle in [23], and will be also used in our denotational model of concurrent programs in Sec. 12.2.2.



Let  $\text{AlpLin-V}$  be an alphabet of **Lingua-V**, i.e., a finite set of characters such that all metaprograms are words over  $\text{AlpLin-V}$ . Let

$$\text{phr} : \text{Phrase} = \text{AlpLin-V}^+$$

be the set of all not empty words over this alphabet that we shall call *phrases*. By a *cut* of a metaprogram

$$\text{mpr} = \text{pre prc} : \text{spr post poc}$$

we mean any pair of phrases *head* and *tail* such that:

- (1)  $\text{head} = \text{pre prc} :$  and  $\text{tail} = \text{spr post poc}$
- (2)  $\text{head} = \text{pre prc} : \text{spr1}$  and  $\text{tail} = \text{spr2 post poc}$  such that  $\text{head} ; \text{tail} = \text{mpr}$
- (3)  $\text{head} = \text{pre prc} : \text{spr}$  and  $\text{tail} = \text{post poc}$

and the semicolon is not “located” in a body of a procedure declarations (we skip a formal definition of “location”). Intuitively, cuts identify such places in the “global” part of a metaprogram, where we can insert

$\text{ins} ;$  in case (1) or  
 $; \text{ins} ;$  in cases (2) or  
 $; \text{ins}$  in case (3)

without violating the syntactic correctness of  $\text{mpr}$ . Note that we do not exclude cuts in class declarations or structured instructions, but we exclude cuts in procedure bodies.

It is evident that cuts of a given metaprogram may be linearly ordered by a relation *earlier/later*. We skip its formal definition. We say that a condition  $\text{con}$  is *satisfied in cut* ( $\text{head}$ ,  $\text{tail}$ ) of  $\text{mpr}$  if the metaprogram:

- (1)  $\text{pre prc} : \text{asr con rsa} ; \text{spr post poc}$  or respectively
- (2)  $\text{pre prc} : \text{spr1} ; \text{asr con rsa} ; \text{spr2 post poc}$  or respectively
- (3)  $\text{pre prc} : \text{spr} ; \text{asr con rsa post poc}$

is correct. Note that in such a case  $\text{mpr}$  must be correct as well.

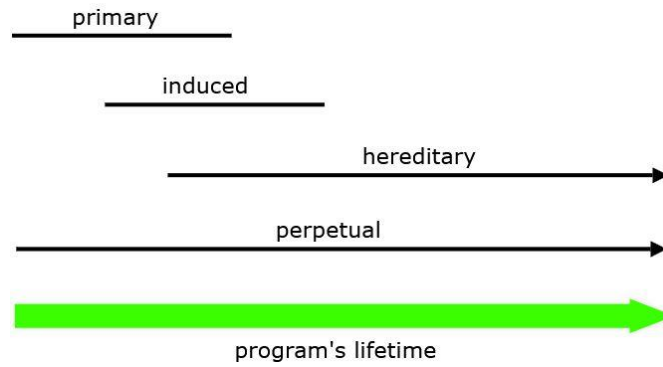
Having defined cuts, we are ready to define *temporal predicates* relative to correct metaprograms. Analogously as in the case of behavioral predicates we define relations between conditions and metaprograms. Let

$$\text{mpr} = \text{pre prc} : \text{spr post poc}$$

be a correct metaprogram. We say that:

|                                   |     |   |
|-----------------------------------|-----|---|
| $\text{con primary in mpr}$       | iff | $\text{con}$ is satisfied at the entrance of the program (and possibly later as well),  |
| $\text{con induced in mpr}$       | iff | there exists a cut of $\text{mpr}$ such that $\text{con}$ is satisfied in this cut; an induced condition must be eventually satisfied,      |
| $\text{con hereditary in mpr}$    | iff | $\text{con}$ once satisfied in a cut, will be satisfied in all later cuts; note that a condition that is never satisfied is hereditary,     |
| $\text{con co-hereditary in mpr}$ | iff | $\text{con}$ once falsified, will be falsified in all later cuts; $\text{con}$ is co-hereditary iff <b>not-k</b> $\text{con}$ is hereditary |
| $\text{con perpetual in mpr}$     | iff | $\text{con}$ is primary and hereditary at the same time, i.e. if it is satisfied in all cuts of $\text{mpr}$ .                              |





**Fig. 9.3-1 Temporal categories of conditions**

The corresponding constructors will be called *temporal metapredicates*.

Note that in all five cases we define a relation between a condition and a correct metaprogram. We do not consider temporal predicates for incorrect metaprograms. To illustrate the introduced concepts let's return to our example of a metaprogram (9.3.4-2). In this metaprogram:

**x is free** — is primary and co-hereditary,  
**var y is integer** — is perpetual, since the type of a variable once declared remains unchanged forever,  
**var x is real** — is induced and hereditary,  
**x = 17,3** — is induced but not necessarily hereditary.

Note that condition **x = 17,3** maybe hereditary or not in this program, depending whether the value of **x** is later modified. The situation with **var x is real with value > 10** is different. It is hereditary in every correct metaprogram since a variable, once declared, can't be redeclared anymore.

### 9.3.6 Language related metaconditions

Behavioral and temporal metaconditions defined so far correspond to relations between conditions and spec- or metaprograms, i.e., are program related. Our next group of metapredicates correspond properties of conditions that are not related to programs, but depend on our language **Lingua-V**. They may be said, therefore, to be *language related*.

**con is immunizing** iff **con hereditary in mpr** for every mpr,  
**con is immanent** iff the value of **con** is never false, although may be undefined or be an error,  
**con is underivable** iff whenever **pre prc : spr pos poc** is correct and  $\text{poc} \Rightarrow \text{con}$ , then  $\text{prc} \Rightarrow \text{con}$ , i.e., must be assumed in **prc** to be satisfied

In **Lingua-V**, typical immunizing conditions are induced by declarations, but if we would allow variable redeclarations, they would not be anymore.

Typical immanent conditions describe mathematical properties of selected sorts of data of our language, such as, e.g.,

**x + y = y + x**

where **+** denotes an integer addition.

A condition is underivable, if it can't be induced in a metaprogram, unless it is metaimplied by the precondition of this program. Typical underivable conditions in **Lingua-V** are conditions of the form **ide is free**<sup>78</sup>. At the same time, since they are essential for declarations, they have to be assumed in the precondition of the program. In

<sup>78</sup> We may also think about "less trivial" underivable conditions such as, e.g., conditions describing properties of databases, that can't be created by **Lingua** programs, but at the same time can be processed by such programs.

practice, in the process of program development, whenever we intend to add a declaration, we have to add an appropriate freeness condition to the precondition of this program, and then to “propagate” it (due to its resilience) to the pre- and postconditions of all preceding programs. More on this issue in Sec. 13.1.

In the process of a metaprogram derivation described in Sec. 9.4.2, underivable conditions and hereditary conditions play contrasting roles:

- Whenever we need an underivable condition in a precondition of an intermediate metaprogram, we have to add it to the precondition of the previous program and, therefore, to the precondition of the initial program.
- Whenever we include a hereditary condition in a postcondition of an intermediate program, we can add it to the postcondition of the next program, and consequently to the postcondition of the final program. We even have to do it, if we want our postcondition to be the strongest one.

As we see, in the process of a metaprogram development we, on one hand, incrementally create the future precondition of this program by adding to it underivable conditions that we shall need later, and on the other — we incrementally create the future strongest postcondition of the program by adding to it hereditary conditions. This means that the strongest postcondition of a program is a conjunction of all its hereditary conditions.

At the same time we have “to keep a repository” of immanent conditions that we shall need to prove some facts, e.g., metaimplications, in developing our metaprograms.

## 9.4 Provably correct metaprograms

### 9.4.1 A deductive perspective on our program development method

As we already mentioned in Sec. 9.1, correct metaprograms may be regarded as valid ground formulas in a theory of denotations, that we have called **D-theory**. From this perspective the development of correct metaprograms is nothing else but proving theorems of this theory. P\In a general case, proving theorems of a formalized theory consist in starting from a set of some “initial theorems” called *axioms*, and the derivation from them new theorems by means of *inference rules*. In the development of theorems of **D-theory** we shall use most frequently use two rules:

#### Substitution rule

*If  $a(x_1, \dots, x_n)$  is a formula with free variables  $x_1, \dots, x_n$ , and  $a(x_1, \dots, x_n)$  belongs to the set of theorems, than for any terms  $ter_1, \dots, ter_n$  of sorts corresponding of variables, the formula  $a(ter_1, \dots, ter_n)$  may be included in the set of theorems (i.e., is a theorem).*

#### Detachment rule (modus ponens)

*For any two formulas  $for_1$  and  $for_2$  if*

- $for_1$  *is a theorem, and*
- $for_1$  *implies  $for_2$  is a theorem*

*then  $for_2$  may be included into the set of theorems (i.e., is a theorem).*

There are also rules for the introduction and elimination of quantifiers, but we shall not discuss them at the moment.

Now, the famous theorem of Austrian mathematician Kurt Gödel (1906 – 1978) insures that all theorems (formulas) derived in this way are valid formulas, i.e., are satisfied for any interpretation of free variables<sup>79</sup>. In

---

<sup>79</sup> Precisely speaking, they are valid in every model of axiom of the theory.

our case this means in particular that all metaprograms developed in our theory are correct. The converse implication for **D-theory** is not true — there exist formulas that are valid but can't be derived (proved)<sup>80</sup>.

For the sake of further considerations formulas of **D-theory** will be called *metaconditions*, and free formulas of the form

**pre** prc : spr **post** poc

will be called the *schemes of metaprograms*. Every metaprogram resulting from a scheme by an application of a substitution rule will be said to be *matching* that scheme. We recall (cf. Sec. 9.1) that free variables of **D-theory** are printed in Arial black.

From this perspective our program-construction rules are valid metaconditions with free variables, which are of one of two following forms:

1. *nuclear rules* — assuring that metaprograms matching certain schemes are correct; they are used in program development process by the application of substitution rule,
2. *implicative rules* — assuring that if some metaconditions are valid, than some metaprograms are correct; they are used in program development process by the application of detachment rule.

An example of a nuclear rule may be the following:

**pre** (ide is free) and-k (tex is type)  
     **let** ide **be** tex **tel**  
**post** var ide is tex (9.4.1-1)

It expresses the fact that for any

ide : Identifier,  
 tex : TypExp,

all metaprograms matching pattern (9.4.1-1) are correct. From this rule we may derive the following concrete correct metaprogram:

**pre** (length is free) and-k (real is type)  
     **let** length **be** real **tel**  
**post** var length is real

In turn, an example of an implicative rule may be the following:

|   |  |
|---|--|
| <b>pre</b> prc : spr <b>post</b> poc<br>poc $\Rightarrow$ poc-1 | <i>metaprogram scheme</i><br><i>metacondition scheme</i> |
| <b>pre</b> prc : spr <b>post</b> poc-1                          | <i>metaprogram scheme</i>                                |

where the vertical arrow denotes the (2-valued) implication in **D-theory**. If this rule is already in the set of our theorems, then in order to apply it in the development of a concrete metaprogram, e.g.,

**pre** x, y is integer and-k x > 1 :  
     y := x+1  
**post** y > 0 (9.4.1-2)

we have to perform the following steps:

First we have to establish a substitution that turns the resulting scheme of the rule into our expected metaprogram. Let's describe it symbolically in the following way:

x, y is integer and-k x > 1     $\rightarrow$  prc  
 y := x+1                         $\rightarrow$  spr

<sup>80</sup> Converse implication is true only for so called first-order theories, where, roughly speaking, variables run over exclusively the elements of a certain universe — e.g., over a set of integers — but not over the subsets of this universe as well. Our **D-theory** does not belong to this category.

|         |                            |
|---------|----------------------------|
| $y > 0$ | $\rightarrow \text{poc-1}$ |
| $y > 1$ | $\rightarrow \text{poc}$   |

Next we have to find in the set of theorems — or to prove — two metaconditions:

```
pre x, y is integer and-k x > 1 :
  y := x+1
post y > 1
```

and

```
y > 1  $\Rightarrow$  y > 0
```

Then, we use the substitution rule to turn the implicative rule (which is a free formula) into a ground metacondition:

```
((pre x, y is integer and-k x > 1 : y := x+1 post y > 1) and (y > 1  $\Rightarrow$  y > 0))
implies
```

(9.4.1-3)

```
(pre x, y is integer and-k x > 1 : y := x+1 post y > 0)
```

where **and** and **implies** are classical connectives of **D-theory**.

Finally, we apply modus ponens to (9.4.1-3) to obtain our target program (9.4.1-2).

To simplify our wording we shall informally identify schemes with syntactic entities that they represent. E.g. we will say that (9.4.1-1) is a metaprogram, rather than a scheme of a metaprogram<sup>81</sup>.

From the perspective of **Lingua-V** our nuclear rule is an atomic metacondition, whereas the implicative rule is a compound metacondition of the form

```
(mco-1 and mco-2) implies mco-3
```

where mco-*i*'s are metaconditions.

It is to be emphasized in this place that in our approach, we do not develop any “logic of programs” as in the approaches of C.A.R. Hoare ( cf. [61], [5], [6]) or E. Dijkstra (cf. [50], [51]), or as in algorithmic logic [10]. Our program construction roles are not (assumed) inference rules of a logic, but theorems proved on the ground of **D-theory**. The axioms of this theory will include two basic categories of axioms:

1. general mathematical axioms like, e.g., axioms of set theory and of number theory,
2. specific axioms about denotations, e.g., all definitions of constructors defined in Sec. 6.

The establishment of axioms of both categories requires a dedicated research and will not be discussed in this book.

A philosophically (but not technically) similar approach is being developed in Dafny project (see [67] - [70]) where the authors talk about programs *correct by construction*. In that project the corresponding language is not given a formal semantics, and consequently, program constructors are not proved sound. The authors tacitly assume that the implementation of their language guarantees the soundness of their program-construction rules.

An ecosystem of programmers in **Lingua-V** should be constructed in such a way that that the only “physical” method of putting a formula into the set of theorems is by applying some inference rules to previously proved theorems.

## 9.4.2 A birds-eye view on a correct metaprogram development

Due to our assumption in Sec. 6.3 every metaprogram is of the form

```
pre prc: spp ; open procedures ; sin post poc
```

---

<sup>81</sup> This convention is analogous to the way we usually talk in intuitive mathematics where we say, e.g., that *p* and *q* are real numbers, rather than that they are variables whose values are real numbers.

where *spp* is a specified program preamble and *sin* is a specified instruction (see Sec. 9.2.6). This form may be “unfolded” to

```
pre prc :
  atp-1 ; ... ; atp-n ; open procedures ; asi-1 ; ... ; asi-k
post poc
```

where

- *atp-i*’s are *atomic preambles*, i.e., single declarations of variables or of classes, or atomic specinstructions,
- *asi-i*’s are *atomic specinstructions*, i.e., instructions listed in Sec. 7.3.6, except the last one, plus assertions; note that structured instructions such as **while-do-od** and **if-then-else-fi** are regarded as atomic, although their “internal instructions” may be quite complex.

Consequently, the process of a correct metaprogram development may be “visualized” as a sequence of steps, each building one *atomic correct metaprogram*:

```
pre prc-1: atp-1 post poc-1
pre prc-2: atp-2 post poc-2
...
pre prc-n : open procedures post poc-n
pre prc-(n+1) : asi-1 post poc-(n+1)
...
pre prc-(n+k+1) : asi-k post poc-(n+k+1)
```

where

```
prc          ⇒ prc-1
poc-i        ⇒ prc-(i+1)  for i = 1,2,...,k+n
poc-(n+k+1) ⇒ poc
```

In this process we are building not only successive imperative components of our future program, but also successive pre- and postconditions. Observe that although the composition of imperative components may be left to the end of the process, the pre- and postconditions have to be built incrementally “as we go”. This means that if starting from some “current metaprogram

```
pre prc-i: atp-i post poc-i
```

(\*)

we create a new metaprogram

```
pre prc-(i+1): atp-(i+1) post poc-(i+1),
```

we have to check if the following metaimplication

```
poc-i ⇒ prc-(i+1)
```

(\*\*)

is satisfied. If that is not the case, one of two actions may be required:

1. the strengthening of *poc-i* to a new one that satisfied (\*) and (\*\*),
2. the modification of (\*) by either modifying *atp-i* or *prc-i* or both; if *prc-i* is modified, the modifications of “earlier” metaprograms may be required.

Note that whatever we want to say about programs must be expressed by derivable conditions or conditions included in the precondition of the program. That is why the derivability, resilience, and heredity of conditions are vital issues in program development.

Since atomic metaprograms will be eventually combined into a final metaprogram using Rule 9.4.3-6 (see later), we may restrict our further considerations to the development of atomic metaprograms. Note in this place that atomic metaprograms do not need to be simple. A metaprogram with a single assignment instruction is simple, but the development of a while loop or of a class declaration with recursive procedures may be far from simple.

In the sequel, we shall define 20+ rules for developing correct metaprograms. Depending on their generality, we split them into two groups:

- *general rules* concerning all metaprograms independent of their content (Sec. 9.4.2),
- *specific rules* concerning declarations (Sec. Sec. 9.4.4 and 9.4.5), and instructions (Sec. 9.4.6).

### 9.4.3 General rules

To emphasize that our construction rules are not axioms but (provable) theorems, we shall call them “lemmas”. However some of these lemmas will have individual names, and then the word “rule” will be used. Our first lemma isn’t in fact a derivation rule, but only an observation.

**Lemma 9.4.3-1** *If a metaprogram*

**pre** prc : spp ; **open procedures** ; sin **post** poc

*is correct and error-sensitive, then in any execution of the included specprogram that starts with a state satisfying prc:*

1. *none of spp, sin, poc generates an error,*
2. *states satisfying prc do not bind identifiers that are declared in spp,*
3. *all active assertions in sin are satisfied,*
4. *the terminal state does not carry an error.*

Intuitively, an assertion is said to be *active* in a specinstruction if it is executed in at least one execution of this specinstruction.

Note that **open procedures** never generate errors, and, therefore, we do not need to mention this fact in our lemma.

**Lemma 9.4.3-2** *If poc is error-sensitive and metaprogram*

**pre** prc : spp ; **open procedures** ; sin **post** poc

*is correct and sin1 has been created from sin by the removal of an arbitrary number of assertions, then the metaprogram*

**pre** prc : spp ; **open procedures** ; sin1 **post** poc

*is correct as well.*

Proof is obvious. The necessity of error-sensitiveness of poc is necessary, since otherwise, if poc is error positive, and we remove an assertion that generates an error, then the postcondition need not to evaluate to tv.

**Lemma 9.4.3-3** *The replacement in a correct metaprograms its pre- or post-condition or a condition in an assertion by a weakly equivalent condition, does not violate the correctness of the program.*

For pre- and post-conditions the proof is obvious. For assertions it follows from the fact that if

$$\text{con1} \Leftrightarrow \text{con2} \quad \text{i.e.} \quad \{\text{con1}\} = \{\text{con2}\}$$

then

$$[\text{con1}].\text{sta} = \text{tv} \quad \text{iff} \quad [\text{con2}].\text{sta} = \text{tv}$$

In particular, this lemma implies that on the level of conditions (but not of boolean expressions of the programming layer!) we can apply all the lemmas of Sec. 9.3.2 that concern weak equivalence.

**Lemma 9.4.3-4** *The replacement in a correct metaprogram of any boolean expression vex in an instruction by a boolean expression vex1 that is strongly equivalent (i.e., such that  $\text{vex} \equiv \text{vex1}$ ) does not violate the correctness of the metaprogram.*

In this lemma we tacitly assume that the relation of strong equivalence of conditions has been extended to value expression. Sun an extension makes sense since strong equivalence simply means the equality of denotations.



**Lemma 9.4.3-5 Rule for sequential composition**

|  |
|--|
| <div style="display: flex; align-items: center;"> <div style="width: 5px; height: 100px; background-color: black; margin-right: 5px;"></div> <div> <pre>pre prc-1: spr-1 post poc-1 pre prc-2: spr-2 post poc-2 poc-1 ⇒ prc-2</pre> </div> </div> <hr style="width: 100%;"/> <div style="display: flex; align-items: center;"> <div style="width: 5px; height: 100px; background-color: black; margin-right: 5px;"></div> <div> <pre>pre prc-1: spr-1;          spr-2 post poc-2 pre prc-1: spr-1; asr poc-1 rsa; spr-2 post poc-2 pre prc-1: spr-1; asr prc-2 rsa; spr-2 post poc-2</pre> </div> </div> |
|--|

Proof is immediate from Lemma 8.7.1-1. Under the line we have a conjunction of metaconditions which means that our lemma represent three single lemmas. The second and the third version will be used in *transformational programming* sketched in Sec. 9.4.6.6.

Our lemma immediately implies the next one:

**Lemma 9.4.3-6 Rule for a final composition**

|   |
|---|
| <div style="display: flex; align-items: center;"> <div style="width: 5px; height: 100px; background-color: black; margin-right: 5px;"></div> <div> <pre>(1) pre prc : spp post (de-con and-k sp-con) (2) pre (de-con and-k sp-con) : open procedures post (de-con and-k op-con and-k sp-con) (3) pre (de-con and-k op-con and-k sp-con) : sin post (de-con and-k op-con and-k si-con)</pre> </div> </div> <hr style="width: 100%;"/> <div style="display: flex; align-items: center;"> <div style="width: 5px; height: 100px; background-color: black; margin-right: 5px;"></div> <div> <pre>pre prc:   spp ; open procedures ; sin post (de-con and-k op-con and-k si-con)</pre> </div> </div> |
|---|

In this rule:

- spp is a specified program preamble,
- de-con is a hereditary condition induced by declarations included in spp,
- sp-con is a condition induced by instructions included in spp,
- op-con is a hereditary condition induced by **open procedures**,
- sin is a specified instruction,
- si-con is a condition induced by sin.

Although our lemma straightforward, we decided to show it, since it illustrates a way in which a final postcondition of a metaprogram is constructed incrementally. This lemma bases on the observation from Sec. 9.3.6 that postconditions induced by declarations are immunizing in **Lingua-V**, and also on an obvious fact that condition sp-con is resilient to **open procedures**. In the third step of our program development, sp-con in the precondition is modified to si-con in the postcondition. This modification describes “the real effect” of the execution of our program.

Note now that whereas an incremental construction of a final postcondition is explicit in our lemma, the process of building precondition is not. It is only implicit in the fact that all declaration-induced conditions — that are necessary to make our programs run cleanly — need some underivable preconditions to be induced. We have to make the latter primitive in our metaprogram, and, of course, they will be temporary. Each of them will cease to be satisfied when it is “consumed” by an associated declaration.

We should emphasize at the end that the three postconditions in our lemma are not necessarily the strongest ones, and therefore are not unique for the involved program. In the process of program development they may be created in many different ways depending on what we want to finally claim about the created program.

**Lemma 9.4.3-7 Rule for strengthening precondition**

|  |
|--|
| <div style="display: flex; align-items: center;"> <div style="width: 5px; height: 100px; background-color: black; margin-right: 5px;"></div> <div> <pre>pre prc : spr post poc prc-1 ⇒ prc</pre> </div> </div> <hr style="width: 100%;"/> <div style="display: flex; align-items: center;"> <div style="width: 5px; height: 100px; background-color: black; margin-right: 5px;"></div> <div> <pre>pre prc-1 : spr post poc</pre> </div> </div> |
|--|



**Lemma 9.4.3-8 Rule for weakening postconditions**

$$\frac{\begin{array}{l} \text{pre } \text{prc} : \text{spr } \text{post } \text{poc} \\ \text{poc} \Rightarrow \text{poc-1} \end{array}}{\text{pre } \text{prc} : \text{spr } \text{post } \text{poc-1}}$$
**Lemma 9.4.3-9 Rule for conjunction and disjunction of conditions**

$$\frac{\begin{array}{l} \text{pre } \text{prc-1} : \text{spr } \text{post } \text{poc-1} \\ \text{pre } \text{prc-2} : \text{spr } \text{post } \text{poc-2} \end{array}}{\begin{array}{l} \text{pre } (\text{prc-1 } \text{and-k } \text{prc-2}) : \text{spr } \text{post } (\text{poc-1 } \text{and-k } \text{poc-2}) \\ \text{pre } (\text{prc-1 } \text{or-k } \text{prc-2}) : \text{spr } \text{post } (\text{poc-1 } \text{or-k } \text{poc-2}) \end{array}}$$
**Lemma 9.4.3-10 Rule for propagation of resilient conditions**

$$\frac{\begin{array}{l} \text{pre } \text{prc} : \text{spr } \text{post } \text{poc} \\ \text{con } \text{resilient to spr} \end{array}}{\text{pre } (\text{prc } \text{and-k } \text{con}) : \text{spr } \text{post } (\text{poc } \text{and-k } \text{con})}$$

The proofs of these lemmas follow immediate from the lemmas 8.7.1-3, 8.7.1-4, 8.7.1-5 and 8.7.1-6 respectively.

**9.4.4 Rules for metadeclarations**

There are five categories of atomic declarations (Sec. 6.7.1) to be considered from the perspective of program-construction rules:

- declarations of variables,
- enrichments of covering relations,
- declarations of classes,
- global openings of procedures.

In all these cases postconditions of corresponding metadeclarations are built in an incremental way. We shall discuss them in the subsequent sections. We have not included in this list declarations of type constants and of methods, since they belong to the category of class transformers (Sec. 6.7.4) and will be discussed later.

The rule for variable declarations is nuclear and is the following:

**Lemma 9.4.4-1 Rule for variable declaration**

$$\frac{\begin{array}{l} \text{pre } (\text{ide is free}) \text{ and-k } (\text{tex is type}) \\ \text{let ide be tex with yex tel} \\ \text{post var ide is tex with yex} \end{array}}{\text{pre } (\text{ide is free}) \text{ and-k } (\text{tex is type})}$$

The proof is obvious. The rule for class attribute declaration is analogous, but belongs to a different category since attribute declarations are executed as components of class declarations; they belong to the category of class transformers.

An enrichment of a current covering relations add new pairs of types to this relations and modifies cov-expression accordingly. This leads us to the following rule:

**Lemma 9.4.4-2 Rule for enrichment of a covering relation**

$$\frac{\begin{array}{l} \text{pre consistent}(\text{tex1}, \text{tex2}) \text{ and-k cov is current:} \\ \text{enrich-cov}(\text{tex1}, \text{tex2}) \\ \text{post } (\text{tex1}, \text{tex2}) ; \text{cov is current} \end{array}}{\text{pre consistent}(\text{tex1}, \text{tex2}) \text{ and-k cov is current:}}$$

The situation with condition **consistent**(tex1 , tex2) is similar to that of **ide is free**. It is not derivable and co-hereditary (Sec. 9.3.1).

Let's consider now a general scheme of a class metadecclaration of the following form:

```
pre prc:
  class
    ide parent cli with
      ctr-1;
    ...
    ctr-k
  ssalc
post poc
```

(9.4.4-1)

where ctr-i's are atomic class transformers and cli is a class indicator which may be an identifier or an empty-class indicator, i.e.:

```
cli : Identifier    or
cli = empty-class.
```

Our goal in the development of this metadecclaration consists in establishing:

- a precondition prc that guarantees a clean execution of our declaration,
- a postcondition poc that describes the effect of this declaration.

To realize this goal let's rewrite (9.4.4-1) to an equivalent form where class transformers are replaced by anchored class transformers:

```
pre prc :
  class
    ide parent cli with skip-ctr
  ssalc ;
  ctr-1 in ide ;
  ...
  ctr-k in ide
post con
```

Given this form we can formulate a scheme of a lemma analogous to the composition lemma in Sec. 9.4.4:

#### Lemma 9.4.4-3 Rule for class declaration

|   |   |
|---|---|
| (1) pre prc<br>(2) pre pa-poc<br>(3) pre (pa-poc and-k cr-poc-1)<br>(4) ...   | : class ide parent cli with skip-ctr ssalc    post pa-poc<br>: ctr-1 in ide    post (pa-poc and-k cr-poc-1)<br>: ctr-2 in ide    post (pa-poc and-k cr-poc-1 and-k cr-poc-2)<br>... |
| pre prc:<br>class ide parent cli with ctr-1; ... ; ctr-k ssalc<br>post pa-poc and-k cr-poc-1 and-k cr-poc-2 and-k ... |   |

The proof of this lemma is immediate from Lemma 9.4.3-5 for sequential composition. What remains to be done now, is to define lemmas for all categories of metadecclarations that may appear above the line.. First of them concerns a declaration of a funding class and is the following:

#### Lemma 9.4.4-4 Rule for a declaration of a funding class

```
pre : (cl-ide is free) and-k (cli is class)
  class cl-ide parent cli with skip-ctr ssalc
post cl-ide child of cli
```

Given this *initiation lemma* we can proceed to the lemmas for anchored class transformers. To save the space (and the resilience of our readers!), we will show only selected examples of such lemmas. We start from the lemma for adding an attribute. It is similar to Lemma 9.4.4-1 for variable declaration.

**Lemma 9.4.4-5 Rule for adding an abstract attribute**

```
pre (at-ide is free) and-k (cl-ide is class) and-k (tex is type) :
  let at-ide be tex with yex as pst tel in cl-ide
post att at-ide is tex with yex in cl-ide as pst
```

**Lemma 9.4.4-6 Rule for adding a type constant**

```
pre (tc-ide is free) and-k (cl-ide is class) and-k (tex is type) :
  set tc-ide be tex tes in cl-ide
post tc-ide is tex
```

**Lemma 9.4.4-6 Rule for adding an imperative pre-procedure declaration**

```
pre (pr-ide is free) and-k (cl-ide is class)
  proc pr-ide (val my-fpc-v ref my-fpc-r) my-body in cl-ide;
post pre-proc pr-ide (val my-fpc-v ref my-fpc-r) my-body imperative in cl-ide
```

The postcondition of the resulting metadecoration has been defined in Sec. 9.2.5. The soundness of this rule is evident from the definition of the applied transformer (Sec. 6.7.4.6). Note that all we need for a pre-procedure declaration to execute cleanly is that its hosting class *cl-ide* has been declared, and its name *pr-ide* is fresh. Lemmas for functional procedures and object constructors are, of course, analogous.

## 9.4.5 The opening of procedures

Our last lemma associated with declarations concerns the global declaration **open procedures**. In this case we can't formulate one universal lemma since the number of class declarations in a program, and the numbers of procedure declarations in each class are unlimited. All we can do, is to formulate the following scheme of a nuclear lemma.

**Lemma 9.4.5-1 Rule for the opening of procedures**

```
pre
  pre-proc pr-ide-11 (val fpc-v-11 ref fpc-r-11) body-11 imperative in cl-ide-1 and-k
  pre-proc pr-ide-12 (val fpc-v-12 ref fpc-r-12) body-12 imperative in cl-ide-1 and-k
  ...
  pre-proc pr-ide-21 (val fpc-v-21 ref fpc-r-21) body-21 imperative in cl-ide-2 and-k
  pre-proc pr-ide-22 (val fpc-v-22 ref fpc-r-22) body-22 imperative in cl-ide-2 and-k
  ...
  open procedures
post
  cl-ide-1.pr-ide-11 opened and-k
  cl-ide-1.pr-ide-12 opened and-k
  ...
  cl-ide-2.pr-ide-21 opened and-k
  cl-ide-2.pr-ide-22 opened and-k
  ...
```

It is to be emphasized that whereas in the precondition we have a conjunction of single conditions, the postcondition is one atomic condition that expresses a property of a tuple of procedures. Such a construction is necessary, since global declarations declare tuples of procedures in “one step”. Therefore, our postcondition should express the fact that procedures assigned to procedure indicators in the current state are identical with procedures created from the corresponding pre-procedures.

## 9.4.6 Rules for metainstructions

### 9.4.6.1 Rules for structured instructions

#### Lemma 9.4.6-1 Rule for conditional branching if-then-else-fi

$$\begin{array}{l} \uparrow \\ \text{pre (prc and-k vex)} : \text{sin1 post poc} \\ \text{pre (prc and-k (not-k vex))} : \text{sin2 post poc} \\ \text{prc} \Rightarrow (\text{vex or-k (not-k vex)}) \\ \hline \text{pre prc : if vex then sin1 else sin2 fi post poc} \\ \downarrow \end{array}$$

Here the two-sided vertical arrow represents two implications: top-to-bottom and a bottom-to-top. The metaimplication above the line guarantees that whenever the precondition is satisfied, the evaluation of boolean expression **vex** terminates, and yields a boolean value rather than an error. Note that in a two-valued logic this meta-declaration would be omitted as a tautology.

The second lemma corresponds to a **while-do-od** loop, where **vex** is a boolean expression, and **inv** is a condition called an *invariant of the loop*:

#### Lemma 9.4.6-2 Rule for loop while-do-od

$$\begin{array}{l} \downarrow \\ (1) \text{ pre (inv and-k vex) : sin post inv} \\ (2) \text{ limited replicability of (asr vex rsa ; sin) if inv} \\ (3) \text{ prc} \Rightarrow \text{inv} \\ (4) \text{ inv} \Rightarrow (\text{vex or-k (not-k vex)}) \\ (5) \text{ inv and-k (not-k vex)} \Rightarrow \text{poc} \\ \hline \text{pre prc : asr inv rsa ; while vex do sin od post poc} \end{array}$$

The metacondition used in (2) has been defined in Sec. 9.3.1. Proof follows directly from lemma Rule 8.7.2-6.

### 9.4.6.2 Rules for assignment instructions

In the case of assignment instructions, instead of formulating a rule “ready to be used”, we show a universal rule and sketch a way of using it in concrete situations. This rule has a tautological character and is the following:

#### Lemma 9.4.6-3 @-tautology

$$\begin{array}{l} \text{pre sin @ con} \\ \text{sin} \\ \text{post con} \end{array}$$

The proof of this rule follows directly from the definition of the denotation of **sin @ con** in Sec. 9.2.7. The idea of using this rule consists in a replacement of the algorithmic precondition by a weakly equivalent one which is not algorithmic. To see, how it works consider as an example the following tautological metainstruction:

$$\begin{array}{l} \text{pre } x := y+1 @ 2*x < 10 \\ \quad x := y+1 \\ \text{post } 2*x < 10 \end{array} \quad (9.4.6.2-1)$$

where we assume that the arithmetical operators **+**, **\*** and **<** are integer operations. It is implicit in this rule that:

- x** has been declared as an integer variable and (therefore) its value is an integer,
- y** analogously,
- y+1** does not generate an error,
- 2\*x** analogously.

Under this assumption we can easily prove the following weak equivalence (note that a strong equivalence does not hold):

$$x := y+1 @ 2*x < 10 \Leftrightarrow (x \text{ is integer}) \text{ and-k } 2*(y+1) < 10 \quad (9.4.6.2-2)$$

Due to our assumptions about operators the left-hand side of the equivalence implies that  $x$  and  $y$  are integer variables. Since on the right-hand side  $x$  does not appear in an expression, we have to add an explicit claim about its type. Note also that the satisfaction of  $2*(y+1) < 10$  means in particular that the expression  $2*(y+1)$  does not generate an 'overload' message.

By Lemma 9.4.3-3, the precondition of (9.4.6.2-1) may be replaced by the right-hand side of (9.4.6.2-2) which leads us to the following metainstruction, which is no more a tautology:

```
pre (x is integer) and-k  $2*(y+1) < 10$ 
   $x := y+1$ 
post  $2*x < 10$ 
```

This step completes the development of a correct metaprogram.

Now, let's apply Lemma 9.4.6-1 in an object-oriented context. Consider the following initial tautological metainstruction:

```
pre  $x.q := y.p.r + 1$  @  $2*x.q < 10$ 
   $x.q := y.p.r + 1$ 
post  $2*x.q < 10$ 
```

where  $x$ ,  $y$  and  $y.p$  point to objects. Now, we can prove the following weak equivalence:

```
 $x.q := y.p.r + 1$  @  $2*x.q < 10$ 
 $\Leftrightarrow$ 
(type of  $x.q$  accepts type of  $y.p.r+1$ ) and-k  $2*(y.p.r+1) < 10$ 
```

Note that it is implicit in (i.e. is metaimplied by) the right-hand side of this equivalence that  $x$ ,  $y$  and  $y.p$  point to objects, and that all involved expressions evaluate cleanly. Basing on this equivalence we can claim the correctness of the following metainstruction:

```
pre (type of  $x.q$  accepts type of  $y.p.r$ ) and-k  $2*(y.p.r+1) < 10$ 
   $x.q := y.p.r + 1$ 
post  $2*x.q < 10$ 
```

### 9.4.6.3 Rules for imperative procedure calls

In a general case, building a program in **Lingua** may be seen as a sequence of the following steps:

- A. given:
  - a. **prc** — a precondition; the postcondition of the preceding program,
  - b. **poc** — a postcondition,
- B. we have to create:
  - a. **rei** — a *reinforcement* of the precondition
  - b. **spr** — a specified program **spr**
- C. such that the following metaprogram is correct:

**pre prc and-k rei : spr post poc** (9.4.6.3-1)

In a general case **rei** may be a conjunction of two conditions:

- **und-rei** — an underivable reinforcement (Sec. 9.3.6), e.g., ensuring the freeness of some identifiers,
- **der-rei** — a derivable reinforcements.

The underivable reinforcement must be conjunctively added to all preconditions and postconditions of the preceding programs. The derivable reinforcement must be insured by appropriate strengthening of the postconditions and preconditions of preceding programs.

Let's analyze now, from this perspective, a special case where our goal is to declare a procedure such that its call has some expected properties. In such a case:

- A. given:

- a. `prc-call` — a precondition,
- b. `poc-call` — a postcondition,

B. we have to create:

- a. `rei` — a precondition reinforcement,
- b. `proc myProc ( val fpa-v ref fpa-r ) begin my-body end` — a procedure declaration (9.4.6.3-2)

C. such that the following metaprogram is correct:

```
pre prc-call and-k rei :
  call MyClass.myProc(val apa-v ref apa-r)
post poc-call. (9.4.6.3-3)
```

D. in the realization of this task, the major subtask and challenge consists in creating a correct metaprogram

```
pre prc-body : my-body post poc-body.
```

that includes the body of our future procedure and — under some conditions — makes (9.4.6.3-3) correct. Let's figure out now what the “some conditions” should be.

In the first place the precondition of the call must guarantee that procedure `myProc` has been declared in `MyClass`, and that it has been opened. This prerequisite may be expressed by two following metaimplication:

```
prc-call ⇒ myProc (val fpa-v, ref fpa-r) begin my-body end imperative in MyClass
prc-call ⇒ procedure MyClass.myProc opened
```

Both conditions in these metaimplications were defined in see Sec. 9.2.5. It is implicit in the first one that `MyClass` has been declared.

Next, `prc-call` must guarantee is that the passing of actual parameters to formal parameters will execute cleanly, and that the resulting state will satisfy `prc-body`. To express this fact we shall use algorithmic condition (5) defined in Sec. 9.2.5, and request the following metaimplication:

```
prc-call ⇒ pass actual val apa-v ref apa-r to formal val fpa-v ref fpa-r with MyClass @ prc-body
```

There is, however, one technical problem hidden in this request. When function `pass-actual` defined in Sec. 6.6.3.4 is generating a local-initial state, this state is getting a declaration-time environment `dt-env` and a call-time store (cf. Sec. 6.6.3.2). In turn, when we evaluate `prc-call` we are dealing with a call-time environment `ct-env`, rather than declaration-time environment `dt-env`. In this place we should recall that `pass-actual` “uses” the environment exclusively to compute the types of formal parameters, and the types declared in `ct-env` are the same as types declared in `dt-env`, which, in turn, is a conclusion of our assumption (see Sec. 6.3) that in programs there are no declarations that would follow the opening of procedures. Consequently, our metaimplication describes adequately our expectations.

The last fact that we have to guarantee, is that the satisfaction of postcondition `poc-body` in a local terminal state `lt-sta` will guarantee:

- a. that the return of references of formal parameters to actual parameters will be executed without an error message,
- b. that after the return of parameters `poc-call` will be satisfied in the global terminal state.

The requirement b. may be expressed by the following metaimplication:

```
poc-body[fpa-r/apa-r] ⇒ poc-call
```

where `poc-body[fpa-r/apa-r]` denotes `poc-body`, where each formal reference-parameter has been replaced by the corresponding actual parameter. Note that for every formal parameter there is exactly one actual parameter (although not necessarily vice versa).

To express requirement a. we have to cope with another technical problem such that the references of actual parameters have to accept the values of corresponding formal parameters in the context of the declaration-time covering relation `dt-cov`, i.e. in the declaration-time environment. By the assumption that all declarations in our programs precede the openings of procedures (Sec. 6.3), and therefore also procedure calls, we can claim that



this relation equals the call-time relation  $\text{ct-cov}$ , but still we have to express a property of a local-terminal state in referring to the  $\text{cov}$ -relation of a “remote” state (see. Fig. 6.6-3). Note in this place that local-terminal relation may be different from (global) call-time relation since an extension of this relation might have taken place in the body of our procedure.

What we have to do in this situation, is to “recall”  $\text{ct-cov}$ , “remembered” in a  $\text{cov}$ -expression  $\text{coe}$  that was adequate at the entrance to the call, i.e., that satisfies metaimplication (Sec. 9.2.3):

$\text{prc-call} \Rightarrow \text{coe}$  **is current**

In practice,  $\text{prc-call}$  will „conjunctively include” condition **coe is current**. Given the call-time  $\text{coe}$  we can request the metaimplication

$\text{poc-body} \Rightarrow \text{fpa-r}$  **accepts**  $\text{apa-r}$  **in**  $\text{coe}$

which means that  $\text{prc-body}$  assures a clean execution of passing reference parameters. Summing up our considerations, we may claim the soundness of the following rule:

#### Lemma 9.4.6-4 Rule for a call of an imperative procedure

- |  |  |
|--|--|
| (1) $\text{prc-call} \Rightarrow \text{myProc}$ ( <b>val</b> $\text{fpa-v}$ <b>ref</b> $\text{fpa-r}$ ) $\text{my-body}$ <b>imperative in</b> $\text{MyClass}$   |  |
| (2) $\text{prc-call} \Rightarrow$ ( <b>pass actual</b> $\text{apa-v}$ <b>ref</b> $\text{apa-r}$ <b>to formal</b> <b>val</b> $\text{fpa-v}$ <b>ref</b> $\text{fpa-r}$ <b>with</b> $\text{MyClass}$ ) <b>@</b> $\text{prc-body}$ |  |
| (3) $\text{prc-call} \Rightarrow$ <b>procedure</b> $\text{MyClass.myProc}$ <b>is opened</b>  |  |
| (4) $\text{prc-call} \Rightarrow \text{coe}$ <b>is current</b>   |  |
| (5) $\text{prc-body} \Rightarrow \text{my-body}$ <b>@</b> $\text{poc-body}$  | i.e. <b>pre</b> $\text{prc-body} : \text{my-body}$ <b>post</b> $\text{poc-body}$ |
| (6) $\text{poc-body} \Rightarrow \text{fpa-r}$ <b>well-valued in</b> $\text{coe}$  |  |
| (7) $\text{poc-body}[\text{fpa-r}/\text{apa-r}] \Rightarrow \text{poc-call}$   |  |

---

**pre**  $\text{prc-call} :$   
   **call**  $\text{MyClass.myProc}$  (**val**  $\text{apa-v}$  **ref**  $\text{apa-r}$ )  
**post**  $\text{poc-call}$

Let us comment this rule.

#### The precondition of the call guarantees that:

- (1) A pre-procedure named  $\text{myProc}$  has been declared in  $\text{MyClass}$  and the denotation of its body is identical with  $[\text{my-body}]$  (cf. Sec. 9.4.6.3). Note that this condition does not say (!) that the body of our procedure is  $\text{my-body}$ .
- (2) The process of passing actual parameters to formal parameters terminates cleanly, and the output state satisfies the precondition of the body.
- (3) Procedure  $\text{MyClass.myProc}$  has been opened, which practically means that the input state is a “follower” of an output state of declaration **open procedures**. Note that in syntactically correct programs metaimplication (3) is a consequence of (1) due to the rule (cf. Sec. 6.3) that no declarations follow **open procedures**. However, we decided to put (3) into our rule, to make it context-independent. In other words, we attempt to express all assumptions necessary for the correctness of our procedure call in terms of the properties of its input states.
- (4) Covering expression  $\text{coe}$  describes adequately the covering relation of the call-time state of the procedure.

#### The precondition of the body guarantees that:

- (5) Every program whose denotation is  $[\text{my-body}]$  — hence, in particular, the body of our procedure — when starting its execution with  $\text{prc-body}$  satisfied, terminates cleanly, and its output state satisfies  $\text{poc-body}$ . Note that  $\text{my-body}$  may include the calls of local and global procedures in which case the proof (generation) of (5) may be far from evident. However, in this rule we only assume that (5) is true (has been generated).

#### The postcondition of the body guarantees that:



- (6) The process of returning formal parameters to actual parameters does not generate an error.
- (7) After the return of the references of formal reference-parameters to actual reference-parameters the post-condition of the call will be satisfied.

#### 9.4.6.4 The case of recursive imperative procedures ???

Tę sekcję trzeba napisać od nowa. Szczególnie przykład.

In Sec. 9.4.6.3 we have introduced a sound construction rule which we can use in the process of building a procedure declaration with expected properties. In this process we have to “invent” such components of a future procedure declaration that the future call of this procedure will be correct for a given postcondition, and an “invented” precondition. Among seven metaimplications that we have to prove in order to ensure the correctness of our call, implication (5) of Rule 9.4.6-4 states that the body of our procedure is correct. In practice, we shall not prove (5) after having developed *my-body*, but we shall develop a correct metaprogram of the form:

```
pre prc-body
  my-body
post poc-body
```

(9.4.6.4-1)

Let’s try to figure out now, what happens, if *my-body* includes a (recursive) call of the procedure, which is being developed? In this case our rule is still adequate, but we can’t establish the correctness of (9.4.6.4-1) without assuming the correctness of the future call:

```
pre prc-call :
  call MyClass.myProc (val apa-v ref apa-r)
post poc-call
```

(9.4.6.4-2)

In other words, in the case of recursion, we can’t first build (9.4.6.4-1) and then claim (9.4.6.4-2), but we have to develop/prove them in a certain sense “in parallel”.

Further on, our procedure may call itself more than once in its body, and not necessarily directly, but also via other procedures. There is, therefore, a potentially infinite number of different mutual-recursion configurations that lead to an infinite number of corresponding construction rules. This situation may be compared to the case of iterative programs with *goto*’s considered in Sec. 8.3. However, whereas in the latter case a way out of the “labyrinth” of a variety of different programming structures was offered by structured programming, an analogous solution for recursive procedures seems doubtful. In the case of recursion, we probably have to treat each recursive structure separately using general rules described in Sec. 8.7.2. We will not delve deeper into this problem, leaving it for further research. Instead, we shall analyze one simple example.

Let *power* be the name of a (future) recursive procedure to be declared in *MyClass*<sup>82</sup>, and let’s assume that our task consists of making the following call correct (we use some obvious colloquializations):

```
pre prc-call
  call MyClass.power(val a,b ref c)
post var a,b,c are integer and-k c=a^b
```

(9.4.6.4-3)

where *prc-call* and the body of our procedure are to be found. As a “candidate declaration” of our procedure declaration let’s take:

```
proc power(val m, n integer ref k integer)
  begin
    if n = 0
      then k := 1
      else n := n-1 ; call MyClass.power(val m, n ref k); k:= k*m
    fi
```

(9.4.6.4-4)

<sup>82</sup> In this case we typeset *power* and *MyClass* in green Arial Narrow, since contrary to the case of Sec.9.4.6.3, now we are talking about a concrete procedure, rather than about a pattern of a procedure.

**end**

Note that this is a declaration of **power** “located” in the declaration of **MyClass** and therefore its global name is **MyClass.power**. As a candidate for the precondition of the call let’s take:

```
prc-call = power (val m, n integer ref k integer) my-body imperative in MyClass and-k
           procedure MyClass.power opened and-k
           var a,b,c are integer with a,b,c ≥ 0
```

where my-body is implicit in (9.4.6.4-4). To prove the correctness of (9.4.6.4-3) we shall use a rule that we recall below:

#### Rule 8.7.2-4 Clean total correctness of a function defined by simple recursion (version 2)

If  $F$  is the least solution of the equation  $X = HXT \mid E$  where  $H$ ,  $T$ , and  $E$  are functions and the domains of  $H$  and  $E$  are disjoint, then the following rule holds:

$$\begin{array}{l} \downarrow \begin{array}{l} (1) (\forall Q) (AQ \subseteq B \text{ implies } A(HQT) \subseteq B) \\ (2) AE \subseteq B \\ (3) A \subseteq FS \end{array} \\ \hline (4) A \subseteq FB \end{array}$$

Where *implies* is the classical 2-valued implication. Now, let:

$A = \{ \text{prc-call} \}$   
 $B = \{ \text{var } a,b,c \text{ are integer with } a,b,c \geq 0 \text{ and-} k \text{ } c=a^b \}$

$H = [ \text{asr } b > 0 \text{ rsa } ] [ b := b-1 ]$   
 $T = [ c := c*a ]$   
 $E = [ \text{asr } b = 0 \text{ rsa } ] [ c := 1 ]$   
 $F = [ \text{call MyClass.power}(\text{val } a,b \text{ ref } c) ]$

To claim (9.4.6.4-3) by means of our rule we have to show in the first place that the denotation of the call of our procedure, i.e.  $F$ , is the least solution of equation

$$F = HFT \mid E$$

where  $H$ ,  $T$  and  $E$  are defined above.

where  $Q$  denotes a denotation of an imperative program-component. We leave the details of this proof to the reader.

#### 9.4.6.5 The case of functional procedures

Analogously to imperative procedures, correctness statements about functional procedures describe the properties of their calls. In this case, however, the result of a call is not a state, whose properties may be described by a condition, but a value. Consider the following (anchored) declaration of a functional pre-procedure:

```
fun funPower(val k, m, n) functional in MyClass
begin
  let k be integer with value ≥ 0 tel
  call MyClass.power(val m, n ref k)
  return 3*k+1
end
```

(9.4.6.5-1)

where **MyClass.power** is the procedure analyzed in Sec. 9.4.6.4. A possible correctness statemet describing a property of of the call may be the following:

```
( funPower (val m, n ref k) begin body return integer end functional in MyClass ) and-k ( a, b ≥ 0 ) :
⇒
call funPower(a, b) = 3*(a^b)+1
```

where **body** is implicit in (9.4.6.5-1). This statement expresses the relationship between the input values of actual parameters **a**, and **b**, and the value exported by the call. Note that we may write our statement in a standard form with pre- and postcondition as:

```
pre funPower (val m, n ref k) begin body return integer end functional in MyClass and-k a, b ≥ 0:
  skip-i
post call funPower(a, b) = 3*(a^b)+1
```

It is implicit in the postcondition that the call evaluates cleanly.

We shall not go into a discussion of building correct functional procedures, leaving it to future research. In the same “spirit” we abandon a discussion of the construction of correct object constructors.

#### 9.4.6.6 Jaco de Bakker paradox in Hoare’s logic

As was noticed by Jaco de Bakker (p. 108, Sec. 4 in [8]) and later commented by K. Apt in [4], on the ground of Hoare’s logic one can prove the formula:

```
pre true : a[a[2]] := 1 post a[a[2]] = 1
```

which for some arrays **a** is not true. Indeed, if

```
a = [2,2].
```

then

```
a[2] = 2
```

hence the execution of the assignment

```
a[a[2]] := 1
```

means the execution of

```
a[2] := 1
```

which means that the new array is **a** = [2,1], and therefore **a**[**a**[2]] = **a**[1] = 2.

Let us observe, however, that Hoare’s problem results neither from having arrays in a language nor from the admission of expressions like **a**[**a**[2]] but from an implicit assumption that whenever such an expression appears on the left-hand-side of an assignment, it should be treated as a variable. As a matter of fact, for many years, programmers used to talk about “subscripted variables” (in Algol 60 [7]) or “indexed variables” (in Pascal [62]).

De Bakker’s problem with Hoare’s logic lies in an imperfect understanding of the meaning (the semantics) of array variables<sup>83</sup>. In our language de Bakker’s paradox does not appear since the instruction of the form:

```
a.(a.2) := 1
```

would be syntactically incorrect. In that place, we write

```
a := change-in-arr a at a.2 by 1 ee
```

or colloquially

```
a := change-in-arr a by a.2 := 1 ee
```

Now, on the ground of our constructions rules we can easily derive the following correct metaprogram:

```
pre a is arr-type number and-k a.1=2 and-k a.2=2
```

<sup>83</sup> In the denotational model described by M. Gordon in [59] array-variables or indexed-variables are admitted on the cost of a rather substantial complication of the model by distinguishing between left-values of expressions (locations) and right-values of expressions (values).

```

a := change-in-arr a by a.2 := 1 ee
post a.1=2 and-k a.2=1

```

## 9.5 Transformational programming

### 9.5.1 First example

In the previous section we were dealing with rules allowing to build correct metaprograms out of correct components. This situation was analogous to an assembly line of, e.g., automobiles. In the present section, we shall consider rules to be used in metaprogram transformations, when we optimize or otherwise change program functionality. In the examples that follow, we shall use some rules introduced earlier as well as some others that we will formalize in Sec. 9.5.3. We will also take into account the fact that integers generated by expressions and assigned to variable can't go beyond certain limits, e.g., that they must belong to the interval  $[-2^{31}, 2^{31} - 1]$ .

In our examples we will use the following denotations:

- $M$  denotes the maximal integer representable in the current implementation,
- $\text{nni}(a-1, \dots, a-n)$  is a predicate satisfied if  $a-1, \dots, a-n$  have been declared as integer variables and their values are representable and not negative, i.e.,  $0 \leq a-i \leq M$ .

Under these assumptions consider two metaprograms:

```

pre nni(x, n) and-k (isrt(n)+1)2 ≤ M:
  x := 0;
  while (x+1)2 ≤ n
    do
      x := x+1
    od
  post x = isrt(n)

```

```

pre nni(x, n, m) and-k ((iqt(n,m)+1)*m ≤ M
  x := 0;
  while (x+1)*m ≤ n
    do
      x := x+1
    od
  post x = iqt(n,m)

```

The first program computes an integer square root denoted by  $\text{isrt}(n)$ , the other — an integer quotient denoted by  $\text{iqt}(n,m)$ . Both metaprograms are searching integer-by-integer through the set of nonnegative integers in seeking the expected result. We may say that they are “driven” by the same while-engine, which is the following:

```

P1: pre nni(x, k) and-k k+1 ≤ M:
  x := 0;
  while x+1 ≤ k
    do
      x := x+1
    od
  post x = k

```

Although the correctness of P1 is pretty obvious, let's derive this program formally using our rules. We start from rule 9.4.6-3 to derive the following correct program (we skip the details):

```

O1: pre nni(x, k) and-k k+1 ≤ M:
  x := 0
  post nni(x, k) and-k k+1 ≤ M and-k x ≤ k

```

Next, using rule 9.4.6-2 we derive

```

O2: pre nni(x, k) and-k k+1 ≤ M and-k x ≤ k :
  while x+1 ≤ k
    do
      x := x+1
    od
  post x = k

```

To show this derivation in details assume that *inv* (our invariant) stands for the precondition of O2, i.e. for:

$\text{nni}(x, k) \text{ and-}k \ k+1 \leq M \text{ and-}k \ x \leq k$

Now, to be able to apply rule 9.4.6-2 we have to prove the following metaconditions:

- (1) **pre** *inv* **and- $k$**   $x+1 \leq k : x := x+1$  **post** *inv*
- (2) limited replicability of  $(\text{asr } x+1 \leq k \text{ rsa } ; x := x+1 )$  if *inv*
- (3)  $\text{inv} \Rightarrow \text{inv}$
- (4)  $\text{inv} \Rightarrow x+1 \leq k \text{ or-}k (\text{not } x+1 \leq k)$
- (5)  $\text{inv and-}k (\text{not } x+1 \leq k) \Rightarrow x = k$

Metacondition (4) states that our invariant guarantees the evaluation of expression  $x+1$  without ‘overload’ error till the exit of the loop where its value becomes  $k+1$ . Note that although in our program the value of  $x$  never equals  $k+1$ , the value of expression  $x+1$  at the exit of the loop reaches this level. This observation leads us to an important general conclusion:

*To avoid errors associated with overloads (and similar events), it is not sufficient to set restrictions on values of variables alone; we must also monitor the values of expressions.*

Basing on our observations in the derivation of Q2 we can now insert an assertion block into it thus getting:

```
O3:  pre nni(x, k) and-k k+1 ≤ M and-k x ≤ k :
asr nni(x, k) and-k k+1 ≤ M and-k x ≤ k in
while x+1 ≤ k
do
  x := x+1
od
  rsa
post x = k
```

From now on new or transformed parts of our programs will be printed in red. Given O1 and O3 we can use rule 9.4.3-5 for sequential composition to get P1 enriched by an assertion:

```
P1 bis : pre nni(x, k) and-k k+1 ≤ M:
x := 0;
  asr nni(x, k) and-k k+1 ≤ M and-k x ≤ k in
while x+1 ≤ k
do
  x := x+1
od
  rsa
post x = k
```

This program may be used now to derive a program that computes the integer square root by substituting *isrt(n)* for  $k$  and assuming that  $n$  has been adequately declared:

```
P2:  pre nni(x,n) and-k isrt(n)+1 ≤ M:
x := 0;
asr nni(x, n) and-k isrt(n)+1 ≤ M and-k x ≤ isrt(n) in
while x+1 ≤ isrt(n)
do
  x := x+1
od
  rsa
post x = isrt(n)
```

Observe now that  $n \leq M$  implies  $\text{isrt}(n)+1 \leq M$  provided that  $M > 3$ . Therefor we can simplify the assertion and get

```
P2:  pre nni(x,n) :
```

```

x := 0;
asr nni(x, n) and-k x ≤ isrt(n) in
while x+1 ≤ isrt(n)
do
  x := x+1
od
  rsa
post x = isrt(n)

```

So far our program computes integer square root by using it, which has little practical value. To cope with this problem we will replace  $x+1 \leq \text{isrt}(n)$  by  $(x+1)^2 \leq n$  based on the following conditional strong equivalence and rule 9.4.3-4:

$x+1 \leq \text{isrt}(n) \equiv (x+1)^2 \leq n$  **whenever**  $\text{nni}(x, n)$  **and-k**  $(\text{isrt}(n)+1)^2 \leq M$  **and-k**  $x \leq \text{isrt}(n)$

Note that in this case we have to strengthen the precondition of P2 with  $(\text{isrt}(n)+1)^2 \leq M$ , since  $(\text{isrt}(n)+1)^2$  is the largest value that  $(x+1)^2$  may reach. This leads us to another general observation:

*In the course of program development we may have to make a step back to strengthen the precondition of the program.*

Our target program is, therefore, the following:

```

P3: pre nni(x, n) and-k  $(\text{isrt}(n)+1)^2 \leq M$ :
x := 0;
asr nni(x, n) and-k  $(\text{isrt}(n)+1)^2 \leq M$  and-k x ≤ isrt(n) in
while  $(x+1)^2 \leq n$ 
do
  x := x+1
od
  rsa
post x = isrt(n)

```

It computes the integer square root without using it.

Our final program is meaningful and correct but is slow since the used searching engine is slow. If we want to speed up our program, we have to use a faster while-engine. Let us start, therefore, from a construction of an engine that searches in logarithmic time.

Let  $\text{po2.z}$  denote a condition that is satisfied if  $z$  is a nonnegative power of 2, i.e., if there exists a  $m \geq 0$  such that:

$z = 2^m$

Let  $\text{mag}(k)$  — called *the magnitude of k* — denote a powers of 2 such that

$\text{mag}(k) \leq k < 2 \cdot \text{mag}(k)$

For instance,  $\text{mag}.11 = 2^3$  since

$2^3 \leq 11 < 2^4$

Now, analogously as in the former example, we can derive two following correct metaprograms:

```

Q1: pre nni(x, k, z) and-k  $2 \cdot \text{mag}(k) \leq M$  :
  z := 1;
  asr nni(x, k, z) and-k  $\text{po2.z}$  and-k  $2 \cdot \text{mag}(k) \leq M$  in
    while  $z \leq k$  do  $z := z \cdot 2$  od
  rsa
post nni(x, k, z) and-k  $z = 2 \cdot \text{mag}(k)$  and-k  $\text{po2.z}$  and-k  $z \leq M$ 

```

and

Q2: **pre** nni( $x, k, z$ ) **and-k**  $z = 2^{\text{mag}(k)}$  **and-k**  $\text{po2}.z$  **and-k**  $z \leq M$  :

```

   $x := 0$ ;
  while  $z > 1$ 
  do
     $z := z/2$ ;
    if  $x+z \leq k$  then  $x:=x+z$  else skip-i fi
  od
post  $x = k$  and-k  $z = 1$ 

```

The first metaprogram computes the successive powers of 2 until it reaches  $2^{\text{mag}(k)}$ , and the second returns from  $2^{\text{mag}(k)}$  to 1 through successive powers  $2^m$  and on its way adds these powers of 2 that “fit” in  $k$ . For instance, since

$$11 = 0 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1$$

the second metaprogram, while given  $2^{\text{mag}.11} = 16$ , will perform the following summation

$$8 + 2 + 1 = 11.$$

In this way, the target value of  $k$  is reconstructed in time proportional to logarithmic, compared to a linear time of metaprogram P3.

If we join both metaprograms on the ground of Rule 9.4.3-5, we get our target metaprogram that finds the value of  $k$  in logarithmic time. In the same step, we move the initialization of  $x$  at the beginning of the metaprogram.

Q3: **pre** nni( $x, k, z$ ) **and-k**  $2^{\text{mag}(k)} \leq M$  :

```

   $z := 1$ ;
   $x := 0$ ;
  asr nni( $x, k, z$ ) and-k  $\text{po2}.z$  and-k  $2^{\text{mag}(k)} \leq M$  in
    while  $z \leq k$  do  $z:=2*z$  od;
    while  $z > 1$ 
    do
       $z := z/2$ ;
      if  $x+z \leq k$  then  $x:=x+z$  fi
    od
  rsa
post  $x = k$  and-k  $z = 1$ 

```

Here and in the sequel

**if**  $vex$  **then** **ins** **fi**

means

**if**  $vex$  **then** **ins** **else skip-i fi**

If in Q3 we replace the expression  $k$  by the expression  $\text{isrt}(n)$ , then we have a program that computes  $\text{isrt}(n)$  but refers to it.

Q4: **pre** nni( $x, z, n$ ) **and-k**  $2^{\text{mag}(\text{isrt}(n))} \leq M$  :

```

   $z := 1$ ;
   $x := 0$ ;
  asr nni( $x, z, n$ ) and-k  $\text{po2}.z$  and-k  $2^{\text{mag}(\text{isrt}(n))} \leq M$  in
    while  $z \leq \text{isrt}(n)$  do  $z:=2*z$  od;
    while  $z > 1$ 
    do
       $z := z/2$ ;
      if  $x+z \leq \text{isrt}(n)$  then  $x:=x+z$  fi
    od
  rsa

```



**post**  $x = \text{isrt}(n)$  **and-k**  $z = 1$

To eliminate  $\text{isrt}(n)$  from our program we shall use two strong conditional equivalences:

$$\begin{aligned} z \leq \text{isrt}(n) &\equiv z^2 \leq n && \text{whenever } \text{nni}(z, n) && \text{and-k } 2 * \text{mag}(\text{isrt}(n)) \leq M \\ x+z \leq \text{isrt}(n) &\equiv (x+z)^2 \leq n && \text{whenever } \text{nni}(z, n, x) && \text{and-k } 2 * \text{mag}(\text{isrt}(n)) \leq M. \end{aligned}$$

In this way we get, similarly as in our former example:

Q4: **pre**  $\text{nni}(x, z, n)$  **and-k**  $2 * \text{mag}(\text{isrt}(n)) \leq M$  :

```

  z := 1;
  x := 0;
  asr  $\text{nni}(x, z, n)$  and-k  $\text{po2}.z$  and-k  $2 * \text{mag}(\text{isrt}(n)) \leq M$  in
    while  $z^2 \leq n$  do  $z := 2 * z$  od ;
    while  $z > 1$ 
      do
         $z := z/2$ ;
        if  $(x+z)^2 \leq n$  then  $x := x+z$  fi
      od
    rsa
  post  $x = \text{isrt}(n)$  and-k  $z = 1$ 
```

Now we shall time-optimize our program by restricting the number of performed operations. Let us start from the observation that in each run of the first loop, the program recalculates the value of  $z^2$ , which is not optimal. To speed up Q4 we introduce a new variable  $q$ , and we enrich our program in such a way that the condition  $q = z^2$  is always satisfied. Such a  $q$  will be called a *register identifier* and  $z^2$  — a *register expression*. This technique is discussed in details in Sec. 9.5.3.

Q5: **pre**  $\text{nni}(x, z, n, q)$  **and-k**  $2 * \text{mag}(\text{isrt}(n)) \leq M$  :

```

  z := 1;
  x := 0;
  q := 1;
  asr  $\text{nni}(x, z, n, q)$  and-k  $\text{po2}.z$  and-k  $2 * \text{mag}(\text{isrt}(n)) \leq M$  and-k  $q = z^2$  in
    while  $q \leq n$ 
      do
        off  $z := 2 * z$ ;  $q := 4 * q$  on
      od
    while  $z > 1$ 
      do
        off  $z := z/2$ ;  $q := q/4$  on
        if  $x^2 + 2 * x * z + q \leq n$  then  $x := x+z$  fi
      od
    rsa
  post  $x = \text{isrt}(n)$  and-k  $z = 1$  and-k  $q = z^2$ 
```

Note that the double-use of **off-on** is necessary since each time when the first assignment destroys the satisfaction of  $q = z^2$ , the second recovers it. For better readability of our program we do not “quote” the assertion in the off-on instruction assuming that it is defined by the context. Now we proceed to further transformations:

1. we use the equivalence  $z > 1 \equiv q > 1$  **whenever**  $(z > 0 \text{ and-k } q = z^2)$  to modify boolean expression in the second loop,
2. we introduce two new variables  $y$  and  $p$  with the conditions  $y = n - x^2$  and  $p = x * z$ ,
3. we use the equivalence  $x^2 + 2 * x * z + q \leq n \equiv 2 * p + q \leq y$  **whenever**  $(y = n - x^2 \text{ and-k } p = x * z)$

Using the corresponding transformations, we get the following program

Q6: **pre**  $\text{nni}(z, n, x, q, y, p)$  **and-k**  $2 * \text{mag}(\text{isrt}(n)) \leq M$  :

```

  z := 1;
  x := 0;
```

```

q := 1;
asr nni(z, n, x, q, y, p) and-k po2.z and-k 2*mag(isrt(n)) ≤ M and-k q = z2 in
  while q ≤ n
  do
    off z:=2*z; q:=4*q on
  od
  y := n;
  p := 0;
  asr y=n-x2 and-k p = x*z in
    while q > 1
    do
      off z:=z/2; q:=q/4; p:=p/2; on
      if 2*p+q ≤ y then x:=x+z; p:=p+q; y:=y-2p-q fi
    od
  rsa
rsa
post x=isrt(n) and-k z=1 and-k q=z2 and-k y=n-x2 and-k p=x*z

```

Contrary to the former introduction of a new variable which was clearly justified, now it not quite clear why  $p$  and  $y$  have been introduced. The answer follows from a well-known truth that in programming, like in playing chase, we sometimes have to predict a few moves in advance. These moves are shown a little later.

In the next transformation, we prepare our metaprogram for the removal of variable  $z$ . For that sake, we perform the following changes:

1. we apply the equivalence  $q=z^2 \Leftrightarrow \text{isrt}(q)=z \text{ whenever } z > 0$  to change the assertion,
2. we use the condition  $\text{isrt}(q) = z$  to replace  $z$  by  $\text{isrt}(q)$  everywhere except the left-hand side of the assignment,
3. we make obvious changes based on the equality  $z=1$ .

The resulting metaprogram is the following:

```

Q7: pre nni(z, n, x, q, y, p) and-k 2*mag(isrt(n)) ≤ M :
  z := 1;
  x := 0;
  q := 1;
  asr nni(z, n, x, q, y, p) and-k po2.z and-k 2*mag(isrt(n)) ≤ M and-k isrt(q)=z in
    while q ≤ n
    do
      off z:=2*isrt(q); q:=4*q on
    od
    y := n;
    p := 0;
    asr y = n-x2 and-k p = x*isrt(q) in
      while q > 1
      do
        off z:=isrt(q)/2; q:=q/4; p:=p/2 on
        if 2*p+q ≤ y then x:=x+isrt(q); p:=p+q; y:=y-2p-q fi
      od
    rsa
  rsa
post x=isrt(n) and-k z=1 and-k q=1 and-k y=n-x2 and-k p=x

```

Now observe that in Q7 variable  $z$  does not appear neither in boolean expressions nor on the right-hand sides of assignment that do not change  $z$ . Since we do not care about the terminal value of  $z$ , we can remove that variable from our metaprogram together with the corresponding assignment and conditions<sup>84</sup>. In this way we get:

```
Q8:  pre nni(n, x, q, y, p) and-k 2*mag(isrt(n)) ≤ M :
      q := 1;
      x := 0;
      asr nni(n, x, q, y, p) and-k 2*mag(isrt(n)) ≤ M in
        while q ≤ n
          do
            q:=4*q
          od
          y := n;
          p := 0;
          asr y = n-x2 and-k p = x*isrt(q) in
            while q > 1
              do
                off q:=q/4; p:=p/2 on
                if 2*p+q≤y then x:=x+isrt(q); p:=p+q; y:=y-2p-q fi
              od
            rsa
          rsa
        post x=isrt(n) and-k q=1 and-k y=n-x2 and-k p=x
```

Now we use the equivalence

$x=\text{isrt}(n) \equiv p=\text{isrt}(n) \text{ whenever } p=x$

to modify the postcondition which makes variable  $x$  not necessary anymore. Therefore, we can remove it with all expressions, and assertions, where it appears.

```
Q9:  pre nni(n, q, y, p) and-k 2*mag(isrt(n)) ≤ M :
      q := 1;
      while q ≤ n do q:=4*q od
      y := n;
      p := 0;
      while q > 1
        do
          off q:=q/4; p:=p/2 on
          if 2*p+q≤y then p:=p+q; y:=y-2p-q fi
        od
      post p=isrt(n) and-k q=1
```

In the last step we replace the instruction

$p:=p/2; \text{ if } 2*p+q \leq y \text{ then } p:=p+q; y:=y-2p-q \text{ fi}$

by an equivalent instruction

$\text{if } p+q \leq y \text{ then } p:=p/2+q; y:=y-p-q \text{ else } p:=p/2 \text{ fi}$

As a result, we get the final version of our metaprogram:

```
Q10: pre nni(n, q, y, p) and-k 2*mag(isrt(n)) ≤ M :
      q := 1;
      while q ≤ n do q:=4*q od
```

<sup>84</sup> In our example the removal of  $z$  with the corresponding assignments and assertions is fairly obvious, but its formulation and proof in a general case may be not evident. We leave it, therefore, to a further research.

```

y := n;
p := 0;
while q > 1
do
  q:=q/4;
  if p+q≤y then p:=p/2+q; y:=y-p-q else p:=p/2 fi
od
post p = isrt(n)

```

This program was written by a well-known Norwegian computer-scientist Ole-Johan Dahl in 1970 to be applied in a microprogrammed arithmetical unit of a computer. It is very time-efficient since in a binary arithmetic the multiplications and divisions by 2 or 4, correspond to simple shifts left or right respectively of binary words. And except shifts it uses only addition and subtraction which are also time inexpensive. In the days when microprocessors were not very fast such optimization was worth the effort.

We do not know in what way Dahl has built this program but we may suppose that he performed an optimization similar to ours, although without formalized rules.

Our example shows a certain specific approach to developing some programs with while-loops by building a program in three steps:

1. writing a program-engine that searches through a specific space of data,
2. installing an appliance on that engine which implements the expected functionality,
3. optimizing the program.

As we are going to see in Sec. 9.5.2, our technique may also be used in changing the types of data elaborated by a program.

## 9.5.2 Changing the types of data

The technique of register identifiers may be also used in the replacement of one data-type by another one. In this section we show how to transform metaprogram Q10 from Sec. 9.5.1 into a metaprogram that operates on binary representations of positive integers. Let

$$\text{bin} : \text{Binary} = \{(0)\} \mid \{(1)\} \odot \{(0), (1)\}^{c*}$$

be the set of binary representations of integers called *binary words*, and let

$$\text{int} : \text{NnInt} = \{0, 1, 2, \dots\}$$

be the set of non-negative integers. We shall use the following functions and relations defined on binary words:

|                                    |               |
|------------------------------------|---------------|
| sl : Binary $\mapsto$ Binary       | shift left    |
| sl.bin =                           |               |
| bin = (0) $\rightarrow$ (0)        |               |
| true $\rightarrow$ bin $\odot$ (0) |               |
| sr : Binary $\mapsto$ Binary       | shift right   |
| sr.bin =                           |               |
| bin = (0) $\rightarrow$ (0)        |               |
| bin = (1) $\rightarrow$ (0)        |               |
| let                                |               |
| (b-1,...,b-(n+1)) = bin            |               |
| true $\rightarrow$ (b-1,...,b-n)   |               |
| + : Binary $\mapsto$ Binary        | addition      |
| - : Binary $\mapsto$ Binary        | subtraction   |
| < : Binary $\mapsto$ {tt, ff}      | less          |
| ≤ : Binary $\mapsto$ {tt, ff}      | less or equal |

The addition and the subtraction of binary words are denoted by the same symbols as for numbers and we assume that they are defined in such a way that the equations (5) and (6) below are satisfied. The orderings are lexicographic and again correspond to their numeric counterparts.

$b2n : \text{Binary} \mapsto \text{NnInt}$   
 $n2b : \text{NnInt} \mapsto \text{Binary}$

binary to number; conversion function  
 number to binary; conversion function

All these functions and relations are defined in such a way that they satisfy the following equations:

- |                              |                         |  |
|------------------------------|-------------------------|--|
| (1) $b2n.(n2b.int)$          | $= int$                 |  |
| (2) $n2b.(b2n.bin)$          | $= bin$                 |  |
| (3) $n2b.(int * 2)$          | $= sl.(n2b.int)$        |  |
| (4) $n2b.(int / 2)$          | $= sr.(n2b.int)$        | where „/” denotes the integer part of division |
| (5) $n2b.(int1 + int2)$      | $= n2b.int1 + n2b.int2$ |  |
| (6) $n2b.(int1 - int2)$      | $= n2b.int1 - n2b.int2$ |  |
| (7) $n2b.int1 < n2b.int2$    | iff $int1 < int2$       |  |
| (8) $n2b.int1 \leq n2b.int2$ | iff $int1 \leq int2$    |  |

Now, we transform metaprogram Q10 by introducing to it three new variables and three corresponding register-conditions:

$Q = n2b(q)$   
 $Y = n2b(y)$   
 $P = n2b(p)$

We assume that a type **binary** has been defined in our language. We introduce the assertions into Q10 and we shift all initialisations to the beginning of our new metaprogram:

**Q11:**    **pre**  $n, q, y, p$  **is**  $nnint$  **and-k**  $Q, Y, P$  **is** **binary** **and-k**  $n \geq 1$   
            $q := 1; Q := (1);$   
            $y := n; Y := n2b(n);$   
            $p := 0; P := (0);$   
           **asr**  $Q = n2b(q)$  **and-k**  $Y = n2b(y)$  **and-k**  $P = n2b(p)$  **in**  
           **while**  $q \leq n$   
           **do**  
           **off**  $q := 4 * q; Q = sl(sl(Q))$  **on**  
           **od**  
           **while**  $q > 1$   
           **do**  
           **off**  $q := q / 4; p := p / 2;$   
            $Q := sr(sr(Q)); P := sr(P);$  **on**  
           **if**  $p + q \leq y$   
           **then off**  $p := p / 2 + q; y := y - 2p - q; P := sr(P) + Q; Y := Y - sl(P) - Q$  **on**  
           **else off**  $p := p / 2; P := sr(P)$  **on**  
           **fi**  
           **od**  
           **rsa**  
           **post**  $p = isrt(n)$  **and-k**  $q = 1$

Now we use four conditional equivalences in order to replace boolean numeric expressions by boolean binary ones:

|                |                           |  |
|----------------|---------------------------|--|
| $q \leq n$     | $\equiv Q \leq n2b(n)$    | <b>whenever</b> $Q = n2b(q)$   |
| $q > 1$        | $\equiv (1) < Q$          | <b>whenever</b> $Q = n2b(q)$   |
| $p + q \leq y$ | $\equiv P + Q \leq Y$     | <b>whenever</b> $Q = n2b(q)$ <b>and-k</b> $Y = n2b(y)$ <b>and-k</b> $P = n2b(p)$ |
| $p = isrt(n)$  | $\equiv P = n2b(isrt(n))$ | <b>whenever</b> $P = isrt(p)$  |

Next we remove from our metaprogram all numeric variables except  $n$  with the corresponding assignments and the assertion block. Since this block reaches the end of the metaprogram, we can modify the postcondition in an appropriate way.

```

Q12: pre  $n \geq 1$  and-k  $Q, Y, P$  is binary and-k  $N = n2b(n)$ ;
 $Q := (1)$ ;
 $Y := N$ ;
 $P := (0)$ ;
while  $Q \leq N$  do  $Q = sl(sl(Q))$  od;
while  $(1) < Q$ 
  do
     $Q := sr(sr(Q)); P := sr(P)$ 
    if  $P + Q \leq Y$ 
      then  $P := sr(P) + Q; Y := Y - sl(P) - Q$ 
      else  $P := sr(P)$ 
    fi
  od
post  $P = n2b(isrt(b2n(N)))$  and-k  $Q = (1)$ 

```

### 9.5.3 Adding a register identifier

This section is devoted to the transformation of metaprograms by adding to them a new identifier  $ide-r$  that satisfies an assertion of the form:

$$ide-r = vex-r.$$

Such transformations were applied in Sec. 9.5.1 in passing from Q4 to Q5 and in Sec. 9.5.2 in passing from Q10 to Q11.

An identifier  $ide-r$  that satisfies the condition  $ide-r = vex-r$  in a certain range will be called a *register-identifier* or just a *register*; the expression  $vex-r$  will be called a *register-expression* and the condition  $ide-r = vex-r$  — a *register-condition*.

Let us start from an obvious generalization of the meaning of  $@$  (Sec. 9.2.2) which now will compose instructions not only with conditions but also with value expressions:

$$[sin @ vex] = [sin] \bullet Sde.[vex]$$

Let's consider a metaprogram that we assume to be correct:

```

P:  pre prc
    sin-h;                                     head (possibly trivial)
    asr con rsa ;
    asr con in ins ; rsa
    sin-t                                     tail (possibly trivial)
post poc

```

Let  $ide-r$  be an identifier which does not appear in  $P$ , and let  $vex-r$  be a value expression such that

**pre** con :  $ide-r := vex-r$  **post** NT

which simply means that **con** guarantees the execution of  $ide-r := vex-r$  without an error or looping. Assume that for states that satisfy **prc**

Under these assumptions a transformation that enriches  $P$  by introducing  $ide-r$  with a register-condition

$$ide-r = vex-r$$

yields a metaprogram:

```

Q:  pre prc and-k  $ide-r$  is tex and-k  $vex$  is tex
    sin-h ;

```

```

    ide-r := vex-r ;
    asr con and-k ide-r = vex-r in $(sin, ide-r = vex-r) rsa
    sin-t
  post poc

```

where  $\$(sin, ide-r = vex-r)$  is defined below in a way which makes  $Q$  correct, provided that  $P$  was correct. The assertion **asr con rsa** has been dropped from  $Q$  (although we could have left it there), since it only served to guarantee, that in its context the value of **vex-r** was defined.

The syntactic operation  $\$$  is defined by structural induction, wrt the structure of **sin**. Let us start from **sin** which is an assignment

```
ide := vex
```

where **ide** is different from **ide-r**, since we have assumed that **ide-r** does not appear in  $P$ .

If **ide** does not appear in **vex-r**, then the execution of this assignment does not cause any change in the value of **vex-r**, and therefore we do not need to add any actualization.

If, however, this is not the case, then directly after **ide:=vex**, we have to add an assignment which recovers the satisfaction of the condition **ide-r = vex-r**. In such a case

```
$(ide := vex , ide-r = vex-r) = off ide := vex; ide-r := vex-r on
```

where equality sign ‘=’ denotes the equality of syntactic elements. An off-clause is necessary here since **ide** appears in **vex-r**. Consequently, the alteration of the value of **ide** may cause the alteration of the value of **vex-r** and the falsification of our condition. In the case of the transformation of  $Q_4$  to  $Q_5$  with a register condition  $q=z^2$  this has led to the enrichment of

```
asr q=z2 rsa ; z:=2*z
```

into:

```
asr q=z2 rsa ; off z:=2*z ; q:=z2 on
```

The assertion has been left in the resulting instruction since we shall need it a little later. Now, our instruction may be changed into an equivalent one (note the inverse order of assignments):

```
asr q=z2 rsa ; off q:=((z:=2*z) @ z2) ; z:=2*z on
```

In this instruction, we can eliminate  $@$ , by transforming the expression  $(z:=2*z) @ z^2$  to a standard form:

```
asr q=z2 rsa ; off q:=4*z2 ; z:=2*z on
```

Now, since the assertion  $q=z^2$  holds “just before” the assignment  $q:=z^2$ , we can replace our instruction by:

```
asr q=z2 rsa ; off q:=4*q ; z:=2*z on
```

which makes the modification of **q** independent of **z**, and therefore — in our example — allows for the elimination of **z** from the metaprogram. In the general case, these transformations are as follows. First the instruction

```
off ide:=vex ; ide-r:=vex-r on
```

is replaced by an equivalent one

```
off ide-r := ((ide := vex) @ vex-r) ; ide := vex on
```

Further on, the expression  $((ide := vex) @ vex-r)$  is transformed to a standard form, and then we try to change it in such a way that the identifier **ide** can be eliminated due to the register-condition **ide-r=vex-r**. This action completes the transformation.

The second “atomic” case to be investigated is a procedure call:

```
call ide(val acp-v ref acp-r)
```

Let us assume that our procedure call appears in a program in the same context as the assignment in the former case. We again have two subcases to be considered.



If none of the actual referential parameters appears in **vex-r**, then we keep the instruction unchanged. In the opposite case, we replace it with the instruction

**off call** ide (**ref** acp-r **val** acp-v); ide-r := vex-r **on**.

This completes the first step of structural instruction. The remaining steps are rather obvious:

```

$((ide-1 ; ide-2), ide-r=vex-r) =
  $(ide-1, ide-r=vex-r) ; $(ide-2, ide-r = vex-r)

$(if vex-b then sin-1 else sin-2 fi, ide-r = vex-r) =
  if vex-b then $(sin-1, ide-r = vex-r) else $(sin-1, ide-r = vex-r) fi

$(while vex-b do sin od, ide-r = vex-r) =
  while vex-b do $(sin, ide-r = vex-r) od

```

In short, after each assignment or a procedure call that changes the value of a register condition, we add a recovering assignment. The generalization of \$ on specinstruction is rather evident.

In the end, let us point out a methodological difference between @ and \$. The former is a character in the syntax of **Lingua-V**, and on the denotational side corresponds to a sequential composition of an instruction denotation with a data-expression denotation. Therefore:

$$\text{Sde}.[\text{sin} @ \text{vex}] = \text{Sin}.[\text{sin}] \bullet \text{Sde}.[\text{vex}]$$

In turn, \$ is a constructor of syntaxes (from the level of **MetaSoft**)

$$\text{\$} : \text{Instruction} \times \text{RegisterCondition} \mapsto \text{Instruction}$$

where

$$\text{RegisterCondition} = \text{Identifier} = \text{ValExp}^{85}$$

It is to be emphasized in the end that an important technical role in the argument about the adequacy of \$ for our goal plays the fact that functional procedures, whose calls may appear in expressions, do not have side effects, i.e., do not modify states.

---

<sup>85</sup> Notice that the first sign of the equality belongs to MetaSoft and denotes the equality of formal languages, whereas the second — typed in **Arial Narrow** — is a character in the syntax of **Lingua**.

## 10 RELATIONAL DATABASES INTUITIVELY

### 10.1 Preliminary remarks

Section 11, which follows this one, is devoted to an extension of **Lingua** by selected database tools offered by SQL (Structured Query Language). Since we don't expect our reader to be familiar with SQL, the present section contains an informal description of some selected SQL-mechanisms that we shall try to formalize later. Several concept that we introduce in both mentioned sections do not appear in standard SQL manuals, and therefore they will be labelled by "(OWN)" which stands for "our OWN notion".

This section refers to several SQL sources since we didn't find a single manual sufficiently complete and unambiguous to identify the meaning of all these SQL mechanisms that we shall talk about. A nice book of Lech Banachowski [9] contains a model of relational databases and a description of SQL standard, but some issues are missing (e.g., three-valued predicates), and some others are only sketched. On the other end of the scale of clarity and preciseness is a thick volume of Paul DuBois [52]. We quote some "definitions" from that book just to show the scale of problems one has to tackle in building a denotational model for SQL. Between these two extremes, but certainly closer to DuBois, are four other books, [54], [60], [79] and [87]. Of course, since all the books mentioned above were published some time ago, certain mechanisms described there may look slightly differ today.

**Lingua-SQL**, whose draft denotational model will be given in Sec. 11, may be regarded as a sort of an API (Application Programming Interface) or a CLI (Call Level Interface)<sup>86</sup> on the ground of **Lingua**. API's have been created for such programming languages as C, PHP, Perl, Python, and CLI's — for ANSI, C, C#, VB.NET, Java, Pascal, and Fortran<sup>87</sup>. In each of these cases, a language is equipped with mechanisms allowing to run functionalities of an existing SQL engine. In our case the situation is different. If in the future anyone would undertake the challenge of implementing **Lingua-SQL** they should first implement their own SQL engine adequate to our denotational model.

Similarly, as in the case of **Lingua**, we shall not attempt to define a "completed" language. We only formalized some selected SQL tools to sketch a denotational framework where a more complete SQL engine might be defined. Our **Lingua-SQL** is even "less complete" than **Lingua**. Whereas in the latter, we tried to include the most universal programming tools, in **Lingua-SQL**, we shall omit many mechanisms that SQL experts might consider essential. Maybe some of our readers will undertake in the future the challenge of filling in the gaps left by us.

### 10.2 Basic values and their types

Types of only one category — table types — appear explicitly in SQL-manuals known to us. Several other types are present only implicitly. They include *basic types*<sup>88</sup> (OWN), i.e., the types of *basic values* (OWN) that appear in the fields of database tables, and *structural types* (OWN) such as the types of columns, rows and databases.

We shall define basic values as pairs consisting of a *basic data* (OWN) and a basic type. Basic data constitute probably one of the least standardized areas of SQL. Their categories may differ not only between different applications but also between different implementations of the same application.

---

<sup>86</sup> CLI refers to the standard ANSI SQL (see [87] p. 359)

<sup>87</sup> Access has not been mentioned on these lists since it is available only together with Microsoft Basic Access.

<sup>88</sup> Some basic data are simple data, as defined in Sec. 4.1, but some others are not.

In the present section, we base mainly<sup>89</sup> on [87], whose authors declare the compatibility with the standard ANSI SQL-2011<sup>90</sup>. The SQL syntax is printed, as in the former parts of the book, in **Arial Narrow**.

Database tables may be regarded as two-dimensional arrays carrying in their fields four sorts of basic values, each of them, except booleans, further split into several subcategories:

- **Numbers** split into two subsorts: *integers* and *decimal numbers* that split further into several types differing from each other by the range of values, e.g., **SMALLINT**, **BIGINT** or **DECIMAL(p, s)**, where **p** (precision) denotes the maximal number of digits and **s** (scale) — the maximal number of digits after the decimal point.
- **Logical values** are handled as in the three-valued predicate calculus of Kleene (Sec. 2.10), and in [87] they are denoted by **TRUE**, **FALSE**, and **NULL** whereas in [54] by **0**, **1**, and **NULL**. Sometimes, e.g., in [60] instead of **NULL** we have **UNKNOWN**.
- **Strings** are, in principle, texts in our sense, but, similarly to numbers, they are split into subtypes depending on a maximal accepted number of characters. For instance, **CHARACTER(n)** is the type of words of the length **n**. The type of a string with varying length limited to **n** is called in [87] **CHARACTER VARYING(n)**, and the type of a string of an unlimited length (whatever it means) is called **BLOB**. There exist also binary strings, and text-strings called **TEXT**.
- **Times** are tuples of three types: **DATE** — (year, month, day), **TIME** — (hour, minute, second), **DAYTIME** — (year, month, day, hour, minute, second).

Although it is nowhere explicitly said, one may guess (cf. [87]) that all sorts of data contain **NULL** that in some context plays the role of an abstract error. The majority of constructors, except boolean constructors, seem to be transparent for that error.

The constructors of basic data may be split into five following groups<sup>91</sup>:

1. Arithmetic operations: **+**, **-**, **\***, **/**.
2. String operations: **CONCAT**, **UPPER**, **LOWER**, **SUBSTR**, **LENGTH**.
3. Time operations: **GETDATE**, **DAYNAME**, **DAYOFMONTH**,
4. Basic predicates: **=**, **<>**, **<**, **<=**, **>**, **>=**, **IS NULL**, **BETWEEN**, **LIKE**.
5. Logical connectives: **NOT**, **OR**, **AND**.

The first group seems apparently quite obvious, but after a closer analysis we may find that it is obvious only in typical situations. E.g.  $2+3 = 5$ , but if we try to add a number to a string (which is possible!), or to add two numbers whose sum exceeds the maximal allowed value, then the expected result is not clear. The source [87] does not comment on such cases at all, and in [52] p. 786, we can read the following<sup>92</sup>:

*If we do not provide (...) correct values to functions, we should not expect reasonable results.*

In another place of the same manual (p. 754) we read:

*(...) expressions that contain big numbers may exceed the maximal range of 64-bits computations in which case they return unpredictable values (our emphasis).*

We noticed that in the definitions of arithmetic operations, **NULL** does not appear, although it could be used as an abstract error. In this place, the worst possible solution has been chosen: instead of an error message, we have an “unpredictable result” which means that the computation does not abort, but generates a false result of an unpredictable value without warning the user.

<sup>89</sup> „Mainly” but not „totally” since this manual also contains gaps.

<sup>90</sup> ANSI is an acronym of American National Standard Institute, and SQL-2011 is a standard accepted by ANSI in December 2011.

<sup>91</sup> The descriptions of 1 to 4 are from [87] (pp. 129 and 180) and of 5 and 6 from [60] (pp. 191 and 201). The terminology is ours.

<sup>92</sup> Our own translation from the Polish version of the book.

Especially many unclarities are associated with default rules for type-conversion. For instance ([52] p. 753) the following rule concerns the addition operation if its arguments are words:

*... ‘+’ is not an operator for the concatenation of texts, as it is the case in some programming languages. Instead, before the performance of the operation, textual strings are converted into numbers. Strings that do not look like numbers (our emphasis) are converted to 0.*

This rule is illustrated with the following examples:

`‘43bc’ + ‘21d’ = 64`

`‘abc’ + ‘def’ = 0`

It hasn’t been explained, if, e.g., `‘43ab2c’` “looks like a number”, and if it does, is it converted to `43` or `432`? It has not been explained either, whether these rules apply to other arithmetic operations.

Fortunately [87] treats conversion a little more seriously — although still informally — introducing four types of conversions:

1. strings to numbers,
2. numbers to strings,
3. strings to dates and times,
4. dates, and times to strings.

String-operators offer fewer ambiguities, but still are defined only for typical situations. For instance, we did not find information about what happens if the concatenation of two strings exceeds an accepted length.

Time-operators offer further examples of inconsistencies between different SQL-applications that concern both the syntax and the types of operators. We shall not further analyze this problem since the involved operators are easy to formalize, once we decide about their meanings.

Predicates are typologically ambiguous since, in the majority of cases, they apply to all four sorts of data. E.g., the operators `‘=’` and `BETWEEN` may be used for numbers and strings and probably also for dates. Their definitions are rather vague. E.g., in [87] p. 130, we can read:

*If in a query, we use the (=) operator, the compared values must be identical, and in the opposite case, the condition is not satisfied.*

It has not been explained if “not satisfied” means “false” or “not true”. E.g. should we regard the value of the boolean expression `12 = abc` as false or error?

The operator `BETWEEN` takes three arguments and checks if the first is between the second and the third in some default ordering.

The operator `LIKE` takes two string-arguments and checks if the first coincides with the pattern described by the second. Patterns are described using letters, digits and two special symbols:

- `%` — an arbitrary string of characters (possibly empty)
- `_` — an arbitrary character

The only source where we found complete definitions of logical operators is [60], where a table-definition is given on page 191 and corresponds to Kleene’s operators defined in Sec. 2.10. It seem rather strange that although we have a `NOT` operator in the language, special negated versions are introduced for all predicates, e.g., `NOT NULL` or `NOT BETWEEN`.

As the above examples illustrate, for all non-boolean operators, we have in SQL a situation which is typical for software manuals. Within the area of standard ranges of arguments, everything is clear. If, however, we go beyond this area, we can hardly predict what will happen. With a high degree of certainty, we may expect to encounter a different surprise in each implementation.

### 10.3 Creating tables

A central SQL-concept is a *table*, that is a two-dimensional array, but may be also regarded as a tuple of named columns where columns are tuples of basic values of a common type. Alternatively, a table may be regarded as a tuple of rows, where rows are mappings from identifiers (column names) to basic values. The intersections of rows and columns are called *table fields*. Of course, in reachable tables all rows have a common set of column names and tables are rectangular.

Tables in SQL are storable, i.e., assignable to variables in memory stores. In the sequel, variables carrying tables will be called *table variables* (OWN) or *table names*. To declare a table variable, we use operator **CREATE TABLE** that assigns to a variable identifier a *table type* and (we can guess) a one-row table of default values indicated by table type<sup>93</sup>.

A *table type* (OWN) may be seen as a tuple consisting of:

- a mapping assigning to each column name a column type,
- a predicate, called a *row yoke* (OWN), that describe a common property of all rows.

In turn, a *column type* (OWN) consist of:

- a basic type to be the common type of all values standing in the column,
- an (optional) default value or an indication that it can't be NULL,
- a predicate, called *column yoke* (OWN), that describes properties of the column,
- a finite set of *marks* (OWN) that describe relationships between tables in the database (Sec. 10.4).

Here is an example of two such declarations cited with only minor modifications after [9] p. 14<sup>94</sup>:

```
CREATE TABLE Affiliations
(
  Department_ID      Number(3)          PRIMARY KEY,
  Department          Varchar(20) NOT NULL UNIQUE
  City               Varchar(50)
);

CREATE TABLE Employees
(
  Employee_ID        Number(6)          PRIMARY KEY,
  Name               Varchar(20) NOT NULL,
  Position           Varchar(9)  DEFAULT NULL,
  Manager            Number(6) ,
  Employment_date    Date,
  Salary             Number(8,2),
  Bonus              Number(8,2),
  Department_ID      Number(3)          REFERENCES Affiliations,
  CHECK(Bonus + Salary < 10000)
)
```

The tabulation in this example shows a certain universal structure of a declaration:

<sup>93</sup> We did not find in the literature on SQL any information about what category of tables is assigned to a table-variable by its declaration.

<sup>94</sup> In Sec. 11 we shall frequently refer to this example and also to some other examples from [9]. In all cases we keep the original notation, where Number(p) denotes a type of total numbers with p digits, and Number(p, s) denotes the type of decimal numbers of the total number of digits equal to p and the number of digits after decimal point equal to s. In turn Varchar(n) denotes the type of strings of length not exceeding n.

- in the first column we see column names<sup>95</sup> of the future table; they will be common to all rows constituting this table,
- the remaining columns carry informations about data stored in table columns; in our model, they will be expressed by the mentioned already four components of a column type (some of them may be optional),

Special cases represent informations expressed by **REFERENCES Affiliations** and **PRIMARY KEY** that indicate a subordination relation between tables (Sec. 10.4),

In the last row of the second declaration we see a row-yoke expression describing the requirement that the values of the fields **Salary** and **Bonus** in each row of the future table satisfy the indicated condition.

The elements of a table declaration, except column names, will be referred to as *integrity constraints of a table*. Their meanings are following:

1. **Number(3)** — the type of data in the column.
2. **DEFAULT** — a default value follows this keyword.
3. **NOT NULL** — all fields in the column must not be empty, i.e., none of them may be **NULL**. An attempt of a violation of this constraint should generate an error signal.
4. **UNIQUE** — no two identical data may appear in the column. If this happens, an error message should be raised.
5. **PRIMARY KEY** — carries two pieces of information: (a) that this column may be a parent column for a column of another table (see Sec. 10.4), (b) that repetitions of elements are not allowed.
6. **REFERENCES Affiliations** — the field **Department\_ID** in table **Employees** is related to the field of the same name in the table **Affiliations**. Relations between tables are used to modify tables and to create queries.
7. **CHECK(Bonus + Salary < 10000)** — all rows of the table should satisfy this condition.

As we see from this example, when we declare a table variable, we define its type<sup>96</sup>. In this way we define seven groups of properties of a future table:

1. the names of columns, e.g., **Department\_ID**,
2. the types of data in all fields of a given column, e.g., **Number(6)**,
3. the default value for a given column, e.g., **DEFAULT NULL**,
4. restrictions concerning columns as a whole (column yokes), e.g., **NOT NULL** or **UNIQUE**,
5. indicator of a special role of a column in the database, e.g., **PRIMARY KEY**,
6. relationships between tables by indicating related columns in tables, e.g., **REFERENCES Affiliations**.
7. relationships between values in each row, e.g., **CHECK(Bonus + Salary < 10000)**; in our model it will be called a *row yoke* (Sec. 11.2.3).

## 10.4 Databases and subordination relation between tables

By a *database* we shall mean a finite collection of named tables, i.e. a mapping from identifiers to tables. A *subordination relation* in a database may be defined as a set of triples of the form

(table name 1, column name, table name 2)

including two (different) names of tables in the base and a common name of their columns. Denotationally these relations may be regarded as yokes that define properties of databases.

<sup>95</sup> It may be slightly misleading at the beginning that in the syntax of table declaration column names are positioned vertically in one column rather than horizontally as they will appear in tables displayed on monitors. It is only a notational convention that facilitates writing table-type declaration with numerous column names.

<sup>96</sup> It seems that SQL lacks a mechanism that would allow to define a table type as a stand-alone element, i.e., independently of a variable declaration.



The mechanism of establishing relations between tables appears in SQL literature in several versions. All of them base on a common idea, although their implementations may be different. Below we try to describe this common idea.

Consider tables **Affiliations** and **Employees** from Sec. 10.3. In **Employees**, we have a column **Department\_ID** which defines the association of an employee to a department. In its declaration we have marking **REFERENCES Affiliations** expressing the fact that in the table **Affiliations** we may find information about the department where an employee is employed. Instead of storing in the table **Employees** the information about the departments where they works, we only show the ID's of these departments that identify appropriate rows in the table **Affiliations**. Now, for this construction to have a practical sense, our two tables must satisfy three conditions:

1. the column **Department\_ID** must appear in both tables,
2. every ID of a department which is in the table **Employees** must also appear in the table **Affiliations** (but not necessarily vice versa),
3. in **Affiliations** the column name **Department\_ID** must have no repetitions.

If these conditions are satisfied, then we say that:

*the column name **Department\_ID** links the tables **Affiliations** and **Employees** with a subordination relation.*

In the pair of tables, **Affiliations**, and **Employees**, the table **Affiliations** is called a *parent* table or a *superior table*, whereas **Employees** is a *child* table or a *subordinated* table. The column name **Department\_ID** is a *primary key* in **Affiliations** and a *foreign key* in **Employees**.

If an employee's row **ER** and a department's row **DR** have the same value in the field **Department\_ID**, then we say that the **ER** *points* to the **DR** (OWN).

The establishment of a subordination relation between tables has consequences for operations on these tables. For instance:

- Introducing an employee who has been employed in a non-existent department should be impossible, i.e. the database-engine should generate an error message in that case.
- A department's record cannot be removed from a table until there are employees employed in that department. An alternative solution is that all employees of the removed department are automatically removed by the engine; a *cascading* solution.
- One can request the creation of a table with chosen columns that combine information from both linked tables, e.g., with columns **Name**, **Department**, **City**.

## 10.5 Instructions of table modification

Tables that have been declared may be modified by instructions. Below we show a few typical examples:

**Entering a new column to a table:**

```
ALTER TABLE Employees
ADD COLUMN ID_number CHAR(11) DEFAULT NULL
```

We add a column to a table, and we indicate a default value for that column.

**Deleting a column from a table**

```
ALTER TABLE Affiliations
DROP COLUMN Department_ID CASCADE
```

This instruction is executed with the option **CASCADE**, which means that the deletion of a column results in the deletion of all columns in the tables of current database that refer to that column. An alternative option is **RE-strict**, where the instruction is not executed whenever such columns exist in the database.

Notice that the instructions from the group **ALTER TABLE** modify not only the content (the data) of a table but also its type. There are other examples of instructions altering tables ([60] p. 49):



- **ALTER COLUMN** — column-type is modified by **SET DEFAULT** or **DROP DEFAULT**, which sets or drops a default value.
- **ADD** — new constraint is added to an existing column.
- **DROP CONSTRAINT** — the removal of a constraint from an indicated column. For this instruction, **RESTRICT** or **CASCADE** must be set.

Another group of table-modifying instructions change the contents of tables without modifying their type. Some typical examples are:

**The insertion of a new record (row):**

```
INSERT INTO Affiliations  
VALUES (095, 'Marketing', 'London')
```

This instruction may also be written in a form where column names are explicit (cf. [54], p. 73)

```
INSERT INTO Affiliations (Department_ID, Dep_name, City)  
VALUES (095, 'Marketing', 'London')
```

In both examples the row-oriented conditions (the row yokes) have been dropped, which means that they are tautologies. However, if this is not the case, these conditions are not modified by this instruction, i.e., the new row has to satisfy them.

**A conditional modification of data in one column.** E.g., the increase of salaries of all salesmen by 10%:

```
UPDATE Employees  
SET Salary = Salary * 1.1  
WHERE Position = 'salesman'
```

**The removal of all rows that satisfy a given yoke.** E.g., the removal of all employees who have no position:

```
DELETE FROM Employees  
WHERE Position IS NULL
```

A particular situation takes place if we drop a row with a primary key which is a foreign key in a child-table, e.g.:

```
DELETE FROM Affiliations  
WHERE Department = 'production'
```

If in the child table **Employees** the key **Department\_ID** is — as in our case — a foreign key and there exist rows which point to the rows that are supposed to be deleted from **Affiliations**, then the operation is not executed and an error message is generated. However, the operation:

```
DELETE FROM Affiliations  
WHERE Dep_name = 'production' CASCADE
```

will be executed, and additionally, in the table **Employees**, all rows that point to the row, which is deleted from **Affiliations**, are deleted as well<sup>97</sup>.

## 10.6 Transactions

By a *transaction*, we mean a sequence of instructions closed (or not) in some parentheses such as, e.g., **BEGIN TRANSACTION** and **COMMIT TRANSACTION**<sup>98</sup>. A transaction may be equipped with an error recovery mechanism that stops the execution of a transaction whenever:

- the execution would violate integrity constraints, or

---

<sup>97</sup> There is a certain inconsistency in SQL compared with the deletion of columns. In the case of rows option **RESTRICT** is set by the system without the possibility of choosing another option by the user.

<sup>98</sup> These parentheses may differ between applications (some manuals are not mentioning them at all). Here we use the notation of Bena Forty ([54], p. 175) which is a standard for Microsoft SQL Server.

- the execution is not possible, e.g., we search for a non-existing element in a table.

In all such cases, the implementation returns to the initial database state of the transaction that is called the *rollback value of the database*<sup>99</sup>.

The fundamental properties of a transaction, often referred to as **ACID** properties, are Atomicity, Consistency, Isolation, and Durability<sup>100</sup>.

*Atomicity* ensures that a transaction is treated as a single, indivisible unit of work. Either all the changes made by the transaction are committed to the database, or none of them are. If any part of the transaction fails, the entire transaction is rolled back to its previous state.

*Consistency* ensures that a transaction brings the database from one valid state to another. The database must satisfy a set of integrity constraints before and after the transaction. If the transaction violates any of these constraints, it is rolled back to maintain a consistent state.

*Isolation* ensures that the execution of one transaction is isolated from the execution of other transactions. Even though multiple transactions may be executing concurrently, the final result should be as if the transactions were executed serially. Isolation prevents interference between transactions and ensures data integrity.

*Durability* guarantees that once a transaction is committed, its effects are permanent and survive any subsequent failures, such as a system crash. The changes made by a committed transaction are stored in non-volatile memory, typically on disk, so that they can be recovered in case of a system failure.

Five following keywords are used to control the recovery mechanism of transactions in SQL-programs:

|                    |  |
|--------------------|--|
| SAVEPOINT          | — save rollback-value of a database      |
| RELEASE SAVEPOINT  | — delete rollback-value                  |
| ROLLBACK           | — call-of transaction                    |
| IF                 | — a conditional activation of a rollback |
| COMMIT TRANSACTION | — accept transaction.                    |

The instruction

**SAVEPOINT** *savepoint-name*

assigns the actual database to a temporary user-defined database name (variable) *savepoint-name*. The instruction

**RELEASE SAVEPOINT** *savepoint-name*

deletes the variable *savepoint-name* (and its value) from the state. The instruction

**ROLLBACK** *savepoint-name*

brings the database to its rollback-value and deletes the variable *savepoint-name*. This instruction may also appear without a parameter, i.e. a *savepoint-name*, in which case the database is (probably?) rolled back to the value initial of transaction-execution<sup>101</sup>. In such cases, the execution of a transaction should start with a default **SAVEPOINT**, which saves database value to some system variable. It also seems that **ROLLBACK** aborts program execution and generates an error message.

To make the execution of **ROLLBACK** dependent on an error message, one may use the conditional **IF** constructor. Ben Forta ([54] p. 179) shows the following example:

**IF @@ERROR <> 0 ROLLBACK** *savepoint-name*

It is explained there that **@@ERROR** is a system-variable whose value equals 0 if there is no error message, and (we guess) equals some error message (or 1?) in the opposite case.

<sup>99</sup> We must warn the reader that, in all the manuals known to us, transactions are described in an exceptionally unclear and incomplete manner. Therefore, our understanding of this construction is based more on speculation than on facts.

<sup>100</sup> Bing AI generated this description of ACID properties on May 22<sup>nd</sup>, 2025..

<sup>101</sup> The parameter-less version of this instruction appears in the majority of manuals known to us.

This example suggests — although that hasn't been explicitly written — that the condition of **IF** might be of the form

```
@@ERROR = error-message
```

with a specific error message. Such a solution would allow making the execution of **ROLLBACK** dependent on the type of an error.

The execution of **COMMIT** results in saving the result of the transaction and deleting all earlier declared roll-back-variables.

For instance, in a database carrying data of bank accounts in a table:

```
CREATE TABLE Accounts
(
  Balance    Number(10,2)
  Client_ID  Number(4)
  ...
)
```

the transaction that moves 1000 \$ from one account to another may have the following form:

```
BEGIN TRANSACTION
```

```
SAVEPOINT start
```

```
UPDATE Accounts
```

```
SET Balance = Balance - 1000
```

```
WHERE Client_ID = 1250 ;
```

```
IF @@ERROR <> 0 ROLLBACK start ;
```

```
UPDATE Accounts
```

```
SET Balance = Balance+ 1000
```

```
WHERE Client_ID = 1260 ;
```

```
IF @@ERROR <> 0 ROLLBACK start
```

```
COMMIT TRANSACTION
```

The first **ROLLBACK** takes place if there is no customer in the database with ID equal 1250, or if its balance-value is less than 1000. The second **ROLLBACK** is activated if the first is not, but there is no customer in the database with ID equal 1260.

Notice that after the execution of the first **UPDATE**, the actual sum of all deposits is not equal to the bank-balance of deposits, which means that the integrity constraints are violated. The second **UPDATE** “removes” this violation, but if it can't be performed because of the lack of 1260-customer, then the transaction would end with an inconsistent database. The second **ROLLBACK** prevents such a situation.

## 10.7 Queries

*Queries* are used to collect information from one or more tables in the form of a new table. The execution of a query results in the generation of a new table and possibly in displaying it on a monitor. Queries are constructed by several variants of operator **SELECT**. Below a few typical examples:

**The selection of indicated columns of a table:**

```
SELECT Name, Salary, Position
FROM Employees
```

As a result of this query, a monitor displays a three-column table with columns indicated by the parameters of **SELECT**.

**The selection of columns combined with the filtering of rows:**

```
SELECT Name, Salary, Position
FROM Employees
WHERE Department_ID = 10
```

In **WHERE** clause, we may have boolean expressions with operators on basic data described in Sec. 10.2.

Queries may be composed of other queries using operators called by Banachowski [9] *algebraic operators on queries*. These operators may be applied to more than one table. For instance:

```
SELECT Department_ID
FROM Affiliations
EXCEPT
SELECT Department_ID
FROM Employees
```

This query generates a one-column table of the ID's of these departments that appear in the table **Affiliations** but that do not appear in the table **Employees**. i.e., the ID's of departments with no employees.

A specific group of queries allows reaching more than one table. In such a case, we say that queries use the *joins of tables*. Below we see an example of a query that selects data from two tables — **Employees** and **Affiliations**.

```
SELECT Employee_ID, Name, Department_ID
FROM Employees, Affiliations
WHERE Employees.Department_ID = Affiliations.Department_ID
AND Affiliations.City = 'London'
```

This query generates a three-column table where each row contains the ID of an employee, his/her name, and the name of the department where he/she is employed. The condition in **WHERE**-clause is called a *joint predicate*. In our case, it returns only such rows where employees are employed in departments located in London.

In **WHERE**-clauses, we may use boolean expressions exploring basic predicates on basic data (Sec. 10.2), e.g.:

```
SELECT Employee_ID, Name, Salary
FROM Employees
WHERE Salary > 1000 AND Salary <= 2000
```

or set-theoretic operators. For instance, the query:

```
SELECT Employee_ID, Name, Position, Salary
FROM Employees
WHERE Position IN ('cashier', 'salesman', 'manager').
```

generates a table with cashiers, salesmen, and managers. The query:

```
SELECT Employee_ID, Name, Position, Salary
FROM Employees
WHERE Salary > ALL
(
  SELECT Salary
  FROM Employees
  WHERE Position = 'cashier'
)
```

generates a table that shows employees with salaries higher than the salaries of all cashiers. In this case, we have to do with a *nested query*, where the inner **SELECT** generates a column with the salaries of all cashiers. Let us denote:

**sae** : **SalEmp** — the set of values in the column **Salary** of the table **Employees**,

`sac` : `SalCas` — the subset of `SalEmp` that contains the salaries of cashiers,  
`shc` : `SalHigCas` — the subset of `SalEmp` that contains salaries higher than the salaries  
 of cashiers

In this case:

$$\text{SalHigCas} = \{ \text{sae} \mid \text{sae} : \text{SalEmp} \text{ and } (\forall \text{sac} : \text{SalCas}) \text{sae} > \text{sac} \}$$

where  $>$  is a predicate that compares numeric values and generates an error whenever at least one of its arguments is not a numeric value.

The transparency of  $>$  implies that the set `SalHigCas` contains numbers only, although it may be empty. In particular, it is empty if `SalCas` contains at least one not-number.

In no bibliographic sources we found information what happens if inequality `sae > sac` generates an error. Will it interrupt a program and generate an error, or the query will generate some “unexpected” table, maybe empty?

## 10.8 Views

If we want to use a query more than once, we may declare it as a procedure. Such procedures are called *views*. Below we see an example of a view-declaration:

```
CREATE VIEW Officials
(Employee_ID, Name, Salary)
AS SELECT Employee_ID, Name, Salary
FROM Employees
WHERE Position = 'official'
```

This view is named `Officials` and creates a three-column table by selecting columns from `Employees` and rows with ‘official’ that stands in the column `Position`.

Since views are procedures, they have no counterparts in syntax (cf. Sec. 6.6.1). At the syntactic level, we only have *view declarations* `CREATE VIEW` and *view calls* (OWN) that refer to the names of views.

View calls may be used in queries in the same way as tables and, of course, a view is executed in the call-time state rather than in the declaration time state. In SQL-manuals, views are, therefore, referred to also as *virtual tables*. Views may also be called in instructions that create or modify tables. Consider the following view-declaration:

```
CREATE VIEW Salesmen
AS SELECT * FROM Employees
WHERE Department_ID = 20
```

In this declaration, the star “\*” means that we chose all columns, and the number 20 is the ID of the sales department. Calling the view `Salesmen` we can create an instruction that modifies the table `Employees` by increasing the salaries of all salesmen by 10%:

```
UPDATE Salesmen
SET Salary = Salary * 1.1.
```

In the case of using views for the modifications of tables, each SQL engine has its specific restrictions. E.g., MySQL requires that in `SELECT`-clauses, only column names may appear.

A special case are *views with check option* which force the checking of a condition when views are used in instructions. Banachowski [9] shows an example of such a view:

```
CREATE VIEW Employees_on_not-paid_holiday
AS SELECT *
FROM Employees
WHERE Salary = 0 OR Salary IS NULL
WITH CHECK OPTION
```

If this view is used in the instruction:

```
UPDATE Employees_on_not-paid_holiday  
SET Salary = 1000  
WHERE Name = 'Smith'
```

then it is not executed if the salary of Smith is 0 or NULL.

## 11 A DENOTATIONAL MODEL FOR DATABASES: **Lingua-SQL**

### 11.1 **Lingua-SQL** as an enrichment of **Lingua**

We shall build **Lingua-SQL** as an enrichment of **Lingua** by new categories of types, values, denotations and corresponding constructors. However, the algebra of denotations of the new language will not be an algebraic extension of the former algebra, i.e., will not include former carries and constructors plus some new ones (see Sec. 2.12). That is why we use here the term “enrichment” rather than “extension”. Formally, the new algebra will include inherently new carriers and constructors. Some of them will be, in a sense, replicas of the formers, and some others will correspond to “genuinely new” mechanisms coming from an SQL engine.

The borderline between **Lingua** and **Lingua-SQL** will be explicitly seen in the definition of *new states* that will be pairs consisting of:

- a state in the former sense, called a *basic component*,
- a new *SQL component* carrying SQL structured values, i.e., *tables* and *databases*.

Databases will carry tables, and tables will carry basic values. We shall assume that the SQL basic values will constitute subcategories of typed data of **Lingua** and therefore will be generated by (formerly defined) value expressions.

We shall not introduce expressions evaluating to tables or databases and consequently we will not have assignment instructions assigning structured SQL values to variables. Tables and databases will be created by declarations, and further developed, enriched or modified by instructions. They will be assigned to their variables directly, rather than via references.

In the category of specific SQL types we will have three subcategories:

- basic types,
- column types,
- table types.

Basic types and column types will include special predicates called *yokes*. We do not introduce database types since they will be implicit in the types of their tables.

In building **Lingua-SQL** we will ensure the satisfaction of two *adequacy principles* relating our model to “typical implementations” of SQL:

1. whenever a typical implementation raises an error message, our model should guarantee the same,
2. whenever in a typical implementation, “one can’t expect a meaningful result” (cf. Sec. 10.2), our corresponding constructor should raise an appropriate error message.

Similarly, as in the case of **Lingua** (cf. Sec. 3.4), we do not pretend to define a “practical” language. Our goal is to identify selected critical challenges in building a denotational framework for database mechanisms, but certainly not to fully cover the SQL diversity of tools. Consequently, our selection of constructors discussed in the sequel has been dictated by the goal of showing typical examples, rather than creating a practical language.

We may also see our experiment with SQL as a case study of adding inherently new mechanisms to an existing programming language.

For compactness, we shall refrain from discussing the syntax of **Lingua-SQL**. We believe this task should be reasonably evident to readers who have gone through Section 7.



## 11.2 Data, types, values and states

### 11.2.1 Basic data, types and values

The domains of basic data are the following (we assume that  $n, p, s, m$  are non-negative integers):

|                                     |   |   |
|-------------------------------------|---|---|
| $\text{bad} : \text{BasDat}$        | $= \text{Boolean} \mid \text{SmaInt} \mid \text{BigInt} \mid \text{Decimal}(p, s) \mid \text{String} \mid$<br>$\text{Date} \mid \text{Time} \mid \text{DateTime} \mid \{\Omega\}$ | basic data  |
| $\text{bad} : \text{Boolean}$       | $= \{\text{true}, \text{false}\}$   | boolean data  |
| $\text{bad} : \text{SmaInt}$        | $= \dots$   | small integers  |
| $\text{bad} : \text{BigInt}$        | $= \dots$   | big integers  |
| $\text{bad} : \text{Integer}.n$     | $= \dots$   | integers with decimal representations of length $n \geq 1$  |
| $\text{bad} : \text{Decimal}(p, s)$ | $= \dots$   | numbers with decimal representations including $p \geq 2$ digits of which $s \geq 1$ digits follow the colon (hence $p > s$ ) |
| $\text{bad} : \text{ChaData}$       | $= \text{Character}.n \mid \text{CharVar}.n \mid \text{Blob}$   | character data  |
| $\text{bad} : \text{Character}.n$   | $= \text{Sign}^{\text{cn}}$   | words of length $n \geq 1$  |
| $\text{bad} : \text{Sign}$          | $= \{a, b, \dots\}$   | a set of signs  |
| $\text{bad} : \text{CharVar}.n$     | $= \text{Character}.1 \mid \dots \mid \text{Character}.n$   | words of length not larger than $n \geq 1$  |
| $\text{bad} : \text{Blob}$          | $= \text{CharVar}.m$  | where (a large) $m \geq 1$ is a parameter of the model  |
| $\text{bad} : \text{Date}$          | $= \text{Year} \times \text{Month} \times \text{Day}$   |   |
| $\text{bad} : \text{Time}$          | $= \text{Hour} \times \text{Minute} \times \text{Second}$   |   |
| $\text{bad} : \text{DateTime}$      | $= \text{Date} \times \text{Time}$  |   |
| $\text{bad} : \text{Year}$          | $= \{0, \dots, 9999\}$  |   |
| $\text{bad} : \text{Month}$         | $= \{1, \dots, 12\}$  |   |
| $\text{bad} : \text{Day}$           | $= \{1, \dots, 31\}$  |   |
| $\text{bad} : \text{Hour}$          | $= \{0, \dots, 23\}$  |   |
| $\text{bad} : \text{Minute}$        | $= \{0, \dots, 59\}$  |   |
| $\text{bad} : \text{Second}$        | $= \{0, \dots, 59\}$  |   |

We assume that all categories of numbers include non-negative and negative numbers and we skip a discussion on their implementation. The categories of numbers and of character-oriented data should be understood as only examples rather than any proposals of standards. The element  $\Omega$  represents an empty field of a table and is called an *empty data* (OWN).

For simplicity we assume that all basic data except time-and-date data are included in the corresponding domains defined in Sec. 4.1, i.e. in *Integer*, *Real* and *Text*. We also assume that all constructors of simple data are applicable to basic data. In the time-and-date data category, we assume to have some typical constructors, but we shall not define them explicitly. We may regard them as parameters of our model.

Basic types, i.e., the types of basic data are defined by the following equations:

|                                 |  |                     |
|---------------------------------|--|---------------------|
| $\text{bat} : \text{BasTyp}$    | $= \text{BoolTyp} \mid \text{IntTyp} \mid \text{DecTyp} \mid \text{ChaTyp} \mid \text{TemTyp}$             | basic types         |
| $\text{bat} : \text{BoolTyp}$   | $= \{\text{'boolean'}\}$   | boolean type        |
| $\text{bat} : \text{IntTyp}$    | $= \text{SmaIntTyp} \mid \text{BigIntTyp}$   | integer types       |
| $\text{bat} : \text{SmaIntTyp}$ | $= \{\text{'smaint'}\}$  | small-integer types |
| $\text{bat} : \text{BigIntTyp}$ | $= \{\text{'bigint'}\}$  | big-integer type    |
| $\text{bat} : \text{DecTyp}$    | $= \{\text{'De'}\} \times \text{SmaInt} \times \text{SmaInt}$  | decimal type        |
| $\text{bat} : \text{ChaTyp}$    | $= \{\text{'Ch'}\} \times \text{SmaInt} \mid \{\text{'ChV'}\} \times \text{SmaInt} \mid \{\text{'blob'}\}$ | character types     |
| $\text{bat} : \text{TemTyp}$    | $= \{\text{'date'}, \text{'time'}, \text{'datetime'}\}$  | date-and-time type  |

We skip obvious definitions of basic type constructors. The clanning function

$\text{CLAN-bt} : \text{BasTyp} \mapsto \text{Sub.BasDat}$

is defined analogously as in **Lingua**. We assume additionally that  $\Omega$  is of any type, i.e.,  $\Omega : \text{CLAN-bt.bat}$  for any  $\text{bat} : \text{BasTyp}$ . By a *basic value* we mean a basic data and its type:

$$\text{bav} : \text{BasVal} = \{(\text{bad}, \text{bat}) \mid \text{bad} : \text{CLAN-bt.bat}\}$$

Basic values of the form  $(\Omega, \text{bat})$  are called *empty values*. The constructors of basic types are defined analogously to the constructors of simple types (Sec. 4.2). We skip their definitions. For technical convenience we introduce two projection functions:

$$\begin{aligned} \text{type} : \text{BasVal} &\mapsto \text{BasTyp} \\ \text{data} : \text{BasVal} &\mapsto \text{BasDat} \end{aligned}$$

Their definitions are obvious.

## 11.2.2 Columns, their yokes and types

By a *column* we mean a nonempty tuple of basic values of a common type:

$$\text{col} : \text{Column} = \{(\text{bav-1}, \dots, \text{bav-n}) \mid n \geq 1 \text{ and } (\forall i,j)(\text{type.bav-i} = \text{type.bav-j})\}$$

For technical convenience we extend the formerly defined function **type** to columns:

$$\text{type} : \text{Column} \mapsto \text{BasTyp}$$

We do not introduce constructors of columns since we will not need them. By a *column yoke* we mean a predicate on columns:

$$\text{coy} : \text{ColYok} = \text{Column} \mapsto \text{BoolValE}$$

Column yokes may be *simple* or *composed*. Composed yokes are propositional compositions of simple yokes. Propositional connectives in column yokes are Kleene's connectives (Sec. 2.10). Simple column yokes are again split into two subcategories:

- *quantified column-yokes* — describing common properties of all elements of a column; e.g., that all elements are greater than 10,
- *holistic column-yokes* — describing properties of columns “as a whole”; e.g., that a column is ordered increasingly or that it is free of repetitions.

The names of constructors of column yokes will be prefixed by **qcy** for quantified yokes and by **hcy** for holistic yokes. Column yokes will become the denotations of corresponding expressions. Typical examples of column yoke constructors may be the following (in- stands for “integer”):

|                        |                   |                  |                           |
|------------------------|-------------------|------------------|---------------------------|
| <b>qcy-greater-in</b>  | : BasVal          | $\mapsto$ ColYok |                           |
| <b>qcy-less-in</b>     | : BasVal          | $\mapsto$ ColYok |                           |
| <b>qcy-equal</b>       | : BasVal          | $\mapsto$ ColYok |                           |
| <b>qcy-later</b>       | : BasVal          | $\mapsto$ ColYok |                           |
| <b>qcy-nonempty</b>    | :                 | $\mapsto$ ColYok |                           |
| ...                    |                   |                  |                           |
| <b>hcy-ordered-lex</b> | :                 | $\mapsto$ ColYok | lexicographically ordered |
| <b>hcy-unique</b>      | :                 | $\mapsto$ ColYok | no repetitions            |
| ...                    |                   |                  |                           |
| <b>coy-and</b>         | : ColYok x ColYok | $\mapsto$ ColYok | conjunction of yokes      |

Let's see the definition of the first quantified constructor. It builds a yoke which is satisfied if every data in a column is of an integer type and is greater than a given integer:

$$\begin{aligned} \text{qcy-greater-in} : \text{BasVal} &\mapsto \text{ColYok} && \text{i.e.} \\ \text{qcy-greater-in} : \text{BasVal} &\mapsto \text{Column} \mapsto \text{BoolValE} \\ \text{qcy-greater-in.bav.col} = & \\ \text{let} & \\ ((\text{bad-1}, \text{bat}), \dots, (\text{bad-n}, \text{bat})) &= \text{col} \end{aligned}$$

|   |                           |
|---|---------------------------|
| (bad, v-bat)  | = bav                     |
| bat /: IntTyp   | → 'integer type expected' |
| ( $\forall 1 \leq i \leq n$ )(da-less-in.(dat-i, bad) | → tv                      |
| <b>true</b>   | → fv                      |

Here *da-less-in* denotes a comparison relation of integers from Sec. 4.1. The definition of the first holistic constructor is the following:

|                                 |      |
|---------------------------------|------|
| hcy-unique : ↦ ColYok           |      |
| hcy-unique : ↦ Column ↦ BooValE |      |
| hcy-unique().col =              |      |
| no-repetitions.col              | → tv |
| <b>true</b>                     | → fv |

The yoke conjunction we define as follows:

|                              |                                    |
|------------------------------|------------------------------------|
| coy-and : ColYok x ColYok    | ↦ ColYok                           |
| coy-and.(coy-1, coy-2).col = |                                    |
| <b>true</b>                  | → coy-1.col <b>and-k</b> coy-2.col |

where **and-k** is the Kleene's operators (see Sec. 2.10)

To define column types we first introduce a domain of *column markings*

|  |                 |
|--|-----------------|
| com : ColMar = FinSub.(Identifier   {'primary'}) | column markings |
|--|-----------------|

A column marking is a finite, possibly empty, set of identifiers plus possibly the mark 'primary'. We need three constructors to build them:

|                                   |          |
|-----------------------------------|----------|
| com-build-empty :                 | ↦ ColMar |
| com-add-primary : ColMar          | ↦ ColMar |
| com-add-ide : Identifier x ColMar | ↦ ColMar |

We skip their obvious definitions. By a *column type* (OWN) we mean a quadruple

|  |
|--|
| (bat, bav, coy, com) : BasTyp x BasVal x ColYok x ColMar |
|--|

such that

|                |
|----------------|
| type.bav = bat |
|----------------|

The basic value bav is called the *default value* of the column type. By

|              |              |
|--------------|--------------|
| cot : ColTyp | column types |
|--------------|--------------|

we denote the domain of column types. Note that in contrast to **Lingua** types of values, column types include yokes. Column types will be built by only one constructor:

|   |                                 |
|---|---------------------------------|
| ct-create : BasTyp x BasVal x ColYok x ColMar | ↦ ColTypeE                      |
| ct-create.(bat, bav, coy, com) =              |                                 |
| type.bav ≠ bat                                | → 'wrong type of default value' |
| <b>true</b>                                   | → (bat, bav, coy, com).         |

The *clans of column types* are defined as sets of columns:

|                  |              |
|------------------|--------------|
| CLAN-ct : ColTyp | ↦ Set.Column |
|------------------|--------------|

such that (bav-1,...,bav-n) : CLAN-ct.(bat, bav, coy, com) iff:

- |  |  |
|--|--|
| (1) type.(bav-1,...,bav-n) = bat               | — the common type of all values of the column is the type indicated by the column type   |
| (2) if data.bav ≠ Ω, then (∀i)(data.bav-i ≠ Ω) | — if the default value of the column is not empty, then all values of this column must not be empty, although they do not need to be the default values, |

- (3)  $\text{coy}.\text{(bav-1, \dots, bav-n)} = \text{tv}$  — the column satisfies the column yoke,
- (4)  $\text{'primary' : com implies no-repetitions. (bav-1, \dots, bav-n)}$  — there are no repetitions in parent columns

Practical implications of (4) will be explained in Sec. 11.2.5.

In the definitions of future table constructors we shall use the following function to check if a given column is of a given type. Note that this function not only checks the type-correctness of a column, but also identifies categories of type violations whenever they occur. This fact is an important feature of our error detection mechanism.

```

check-column-type : Column x ColTyp  $\mapsto$  {'OK'} | Error
check-column-type.(col, cot) =
  let
    (bav-1, ..., bav-n) = col
    (bat, bav, coy, com) = cot
  type.col  $\neq$  bat  $\rightarrow$  'type inconsistency' for i = 1;n
  data.bav  $\neq \Omega$  and data.bav-i =  $\Omega$   $\rightarrow$  'value must not be empty' for i = 1;n
  coy.col : Error  $\rightarrow$  coy.col
  coy.col = fv  $\rightarrow$  'column yoke not satisfied'
  'primary' : com and are-repetitions.col  $\rightarrow$  'repetitions in a parent column'
  true  $\rightarrow$  'OK'

```

Of course

$\text{CLAN-ct.cot} = \{\text{cod} \mid \text{check-column-type.}(\text{cod}, \text{cot}) = \text{'OK'}\}$

In the sequel we shall use the following auxiliary projection function:

```

marking : ColTyp  $\mapsto$  ColMar
marking.(bat, bav, coy, com) = com

```

### 11.2.3 Row yokes of tables

Similarly to column yokes, also row yokes may be seen as predicates on tuples of values. However, they significantly differ in nature:

- column yokes are assigned to individual columns and describe:
  - either properties of individual column elements common to all elements, e.g., that all dates are in a certain time interval,
  - or global properties concerning a column as a whole, e.g., that its is ordered increasingly or has no repetitions,
- row yokes are assigned to tables and describe common properties of all rows of a table; besides these properties refer to (named) elements of rows and may describe not only individual properties of these elements, but also relationships between them, e.g.,  $\text{CHECK}(\text{bonus} + \text{salary} < 10000)$  (cf. Sec. 10.3).

Since row yokes will refer to column names, to define them we introduce a concept of *labeled rows* (OWN):

$\text{lar: LabRow} = \text{Identifier} \Rightarrow \text{BasVal} \mid \{\Omega\}$  labeled row

Labeled rows will be also used in the definitions of table constructors. *Row yokes* are functions that given a labeled row return a basic value or an error:

$\text{roy : RowYok} = \text{LabRow} \mapsto \text{BasValE}$  row yokes

To make the reachable part of the domain of row yokes rich enough, we allow them to return arbitrary basic values, rather than just boolean values.

Row yokes will become the denotations of row-yoke expressions, and the latter will be similar to simplevalue expressions. In their case, however, column names will stand in the place of variables, and they will point to basic

values directly rather than via references<sup>102</sup>. Constructors of row yokes refer to (call) simple-value constructors and possibly some other special constructors otherwise not available at the level of expressions.

Constructors of row yokes are similar to constructors of value-expression denotations. Primary constructors get a value from a row:

ry-get-val : Identifier  $\mapsto$  RowYok                      i.e.  
 ry-get-val : Identifier  $\mapsto$  LabRow  $\mapsto$  BasValE  
 ry-get-val.ide.lar =  
   lar.ide = ?     $\rightarrow$  'no such column name'  
   lar.ide =  $\Omega$     $\rightarrow$  'value undefined'  
   true            $\rightarrow$  lar.ide

Further examples of constructors refer to addition and comparison of integers:

ry-add-int : RowYok x RowYok  $\mapsto$  RowYok  
 ry-less-int : RowYok x RowYok  $\mapsto$  RowYok

The definition of the first constructor is the following:

ry-add-int : RowYok x RowYok  $\mapsto$  RowYok                      i.e.  
 ry-add-int : RowYok x RowYok  $\mapsto$  LabRow  $\mapsto$  BasValE  
 ry-add-int.(roy-1, roy-2).lar =  
   roy-i.lar : Error     $\rightarrow$  roy-i.lar                      for i = 1,2  
   **let**  
     bav-i              = roy-i.lar                      for i = 1,2  
     (dat-i, typ-i) = bav-i                      for i = 1,2  
   typ-i /: IntTyp     $\rightarrow$  'integer expected'    for i = 1,2  
   **true**                $\rightarrow$  td-add-in.(bav-1, bav-2)

In this definition td-add-in is an integer addition of typed data (Sec. 4.3). Note that it may generate an error message.

We shall not define further constructors of row yokes but we assume that they will not generate any “holistic” properties of labelled rows, such as, e.g., that a row is ordered increasingly, or has no repetitions. To express our expectations about row yokes formally, we shall assume that they are *monotone*, in the sense that if a labeled row lar-2 is an extension (as a mapping) of lar-1, then for every monotone yoke roy:

roy.lar-1 = roy.lar-2

## 11.2.4 Tables and their types

Tables are central concept in SQL. Intuitively our tables will consist of a table type and a two-dimensional array of basic values called a *table-content*. However, depending on a table constructor, which we shall apply to a table, we shall regard table content as a tuple of labeled columns called a *column table-content* (OWN) or a tuple of labeled rows called a *row table-content* (OWN). To formalize these two perspectives of seeing a table we introduce two following domains:

ctc : ColTabCon = Identifier  $\Rightarrow$  Column                      column table-contents  
 rtc : RowTabCon = LabRow<sup>c+</sup>                                      row table-contents

By a *rectangular column table-content* we mean a column table-content where all columns are of the same length. In an analogous way we define *rectangular row table-content* and we introduce two corresponding domains:

ctc : ReColTabCon                                      rectangular column table-contents  
 rtc : ReRowTabCon                                      rectangular row table-contents

<sup>102</sup> We do not introduce reference in the SQL-part of our model since database value will not appear in objects. We return to this issue in Sec. 11.2.6.

By the *depth of a column table-content*, in symbols

$\text{depth.ctc}$

we mean the common length of its columns.

By a *table header* (OWN) we mean a nonempty mapping assigning column types to column names. By a *table type* (OWN) we mean a pair consisting of a table header and a row yoke.

$\text{tah} : \text{TabHea} = \text{Identifier} \Rightarrow \text{ColTyp}$

table headers

$\text{tat} : \text{TabTyp} = \text{TabHea} \times \text{RowYok}$

table types

Table types are built by three constructors. The first of them creates a one-column table header.

$\text{create-tab-hea} : \text{Identifier} \times \text{ColTyp} \mapsto \text{TabHea}$

$\text{create-tab-hea}(\text{ide}, \text{cot}) = [\text{ide}/\text{cot}]$

The second constructor adds a new column to an existing table header:

$\text{add-to-tab-hea} : \text{TabHea} \times \text{Identifier} \times \text{ColTyp} \mapsto \text{TabHea}$

$\text{add-to-tab-hea}(\text{tah}, \text{ide}, \text{cot}) =$

$\text{tah.ide} = ! \quad \rightarrow \text{'column name already exists'}$

**true**  $\rightarrow \text{tah}[\text{ide}/\text{cot}]$

The third constructor creates a table type by adding a row yoke to a table header:

$\text{create-tab-typ} : \text{TabHea} \times \text{RowYok} \mapsto \text{TabTyp}$

$\text{create-tab-typ}(\text{tah}, \text{roy}) = (\text{tah}, \text{roy}).$

Given a table type with  $n$  columns:

$\text{tat} = ([\text{ide-1}/\text{cot-1}, \dots, \text{ide-n}/\text{cot-n}], \text{roy})$  where  
 $\text{cot-i} = (\text{bat-i}, \text{bav-i}, \text{coy-i}, \text{com-i})$  for  $i = 1;n$

by the *clan* of this type, denoted by  $\text{CLAN-tt.tat}$ , we mean the set of all rectangular column table-contents with  $n$  columns named by the  $\text{ide-i}$ 's of  $\text{tat}$ , i.e.,

$\text{ctc} = [\text{ide-1}/\text{col-1}, \dots, \text{ide-n}/\text{col-n}]$  where  
 $\text{col-i} = (\text{bav-i1}, \dots, \text{bav-ik})$  for  $i = 1;n$  and some common  $k \geq 1$

such that

- (1) each column is of the corresponding column type, i.e.  
 $\text{col-i} : \text{CLAN-ct.cot-i}$  for  $i = 1;n$ ,
- (2) all labelled rows of the table satisfy the row yoke of the table type, i.e.  
 $\text{roy}[\text{ide-1}/\text{bav-1j}, \dots, \text{ide-n}/\text{bav-nj}] = \text{tv}$  for  $j = 1;k$

By a *table* we shall mean a pair consisting of a rectangular column table-content and its type:

$\text{tab} : \text{Table} = \{(\text{ctc}, \text{tat}) \mid \text{ctc} : \text{CLAN-tt.tat}\}$

tables

Similarly as in the case of column types, also now we define a function that checks if a column table-content is of a given table type:

$\text{type-table-check} : \text{ColTabCon} \times \text{TabTyp} \mapsto \{\text{'OK'}\} \mid \text{Error}$

$\text{type-table-check}(\text{ctc}, \text{tat}) =$

**let**

$[\text{ide-1}/\text{col-1}, \dots, \text{ide-n}/\text{col-n}] = \text{ctc}$

$(\text{bav-i1}, \dots, \text{bav-ik}) = \text{col-i}$

for  $i = 1;n$

$[\text{ide-1}/\text{cot-1}, \dots, \text{ide-n}/\text{cot-n}] = \text{tat}$

$\text{lar-j} = [\text{ide-1}/\text{bav-1j}, \dots, \text{ide-n}/\text{bav-nj}]$

for  $j = 1;k$

$\text{check-column-type}(\text{col-i}, \text{cot-i}) \neq \text{'OK'} \rightarrow \text{check-column-type}(\text{col-i}, \text{cot-i})$

for  $i = 1;n$

$\text{roy.lar-j} : \text{Error}$

$\rightarrow \text{roy.lar-j}$

for  $j = 1;k$

$\text{roy.lar-j} = \text{fv}$

$\rightarrow \text{'row yoke violated'}$

**true**  $\rightarrow$  'OK'

In the sequel we shall use two following functions to “switch” between two perspectives of seeing tables:

**C2R** : ReColTabCon  $\mapsto$  ReRowTabCon column-to-row conversion

**C2R**.[ide-1/col-1,...,ide-n/col-n] =

**let**

(bav-1i,...,bav-ki) = col-i

for i = 1;n

lar-j = [ide-1/bav-j1,...,ide-n/bav-jn]

for j = 1;k

**true**  $\rightarrow$  (lar-1,..., lar-k)

**R2C** : ReRowTabCon  $\mapsto$  ReColTabCon

row-to-column conversion

**R2C**.(rod-1,...,rod-k) =

**let**

[ide-1/bav-j1,...,ide-n/bav-jn] = lar-j

for j = 1;k

col-i = (bav-1i,...,bav-ki) for i = 1;n

**true**  $\rightarrow$  [ide-1/col-1,...,ide-n/col-n]

Of course, each of these function is an inverse of the other, which means that the following functions are identities:

**C2R** • **R2C**

**R2C** • **C2R**

## 11.2.5 Databases and their subordination relations

By a *database* we shall mean a mapping assigning tables to their names:

**dab** : DatBas = Identifier  $\Rightarrow$  Table.

Databases, similarly to tables, belong to the category of values since they combine data and types. The types of databases are implicit in the types of their tables and carry two kinds of information:

- pieces of information about individual tables, saved in the types of columns (except column markings) and in row yokes,
- information about subordination relationships between tables, saved in column markings.

To define the relationships between tables, consider a database **dab**, two identifiers **ch-ide** and **pa-ide** which are the names of two tables in this base and an identifier **co-ide** which is a common name of two columns of these tables. Let further:

|                                  |                        |                    |
|----------------------------------|------------------------|--------------------|
| <b>ch-tab</b>                    | <b>= dab.ch-ide</b>    | child table        |
| (ch-ctc, (ch-tah, ch-roy))       | <b>= ch-tab</b>        |                    |
| <b>ch-col</b>                    | <b>= ch-ctc.co-ide</b> | child column       |
| (ch-bat, ch-bav, ch-coy, ch-com) | <b>= ch-tah.co-ide</b> | child-column type  |
| <b>pa-tab</b>                    | <b>= dab.pa-ide</b>    | parent table       |
| (pa-ctc, (pa-tah, pa-roy))       | <b>= pa-tab</b>        | parent column      |
| <b>pa-col</b>                    | <b>= pa-ctc.co-ide</b> |                    |
| (pa-bat, pa-bav, pa-coy, pa-com) | <b>= pa-tah.co-ide</b> | parent-column type |

We say that a *content-subordination relation* (OWN) holds between our tables via column name **co-ide** that we shall write as

**ch-ide** content-subordinated [**co-ide**] **pa-ide**,

if two following conditions are satisfied:

1. the *child column* **ch-col** contains only such elements that are included in the parent column,
2. the *parent column* **pa-col** is repetition-free,
3. none of these columns include  $\Omega$ .



In such a case we shall say that:

- **ch-tab** is a *child table*, **pa-dab** is its *parent table* and **co-ide** is their *subordination connector*,
- **ch-tab.co-ide** is a *child column*, **pa-tab.co-ide** is its *parent column*,

The conjunction of 1. and 2. implies that each row of the child table unambiguously identifies — *points to* — a row in the parent table. Why we formally define the subordination between the names of tables rather than between the tables themselves will be seen in a moment.

In Fig. 11.2-1, we see an example of two tables (we show only their contents) which are in a content-subordination relation:

Employees content-subordinated [Department] Affiliations

| Employees |              |
|-----------|--------------|
| Name      | Department   |
| Fog       | Distribution |
| Pickwick  | Distribution |
| Weller    | Kitchen      |

| Affiliations |            |
|--------------|------------|
| Department   | City       |
| Distribution | London     |
| Bookkeeping  | Manchester |
| Kitchen      | Edinburgh  |

**Fig. 11.2-1** Employees is a child of Affiliations via Department

Notice that there may be some elements in a parent column that do not appear in the corresponding child column, e.g., departments with no employees. It is also possible that a child column for one table is, at the same time, a parent or a child column for another table, or that it is a child column for more than one table. However, we exclude cases where a table is a child of itself, although cycles are allowed. And, of course, one parent may have many children.

Note now that an existing subordination relation between two tables may be destroyed when we modify one of these tables, e.g. if to a child column we add a value that does not appear in the parent column, or if we remove or change a value in a parent column. To prevent such situations whenever a subordination is critical, SQL offers a column marking mechanism, allowing programmers to mark a column as a child or a potential parent.

We have seen examples of such markings in Sec. 10.3 where:

- child table **Employees** includes column **Department\_ID** marked as a child by **REFERENCES Affiliations**,
- parent table **Affiliations** includes column **Department\_ID** marked as a parent by **PRIMARY KEY**.

To include this mechanism in our model we introduce a second relation between tables called a *marking-subordination relation* written as

ch-ide marked-subordinated [co-ide] pa-ide,

which holds if:

- **pa-ide** : **ch-com**; in this case we say that:
  - column **ch-tab.co-ide** is *marked to be a child* of parent column **pa-tab.co-ide**,
  - table **ch-tab** is *marked to be a child* of parent table **pa-tab**,
- **'primary'** : **pa-com**; in this case we say that:
  - column **pa-tab.co-ide** is *marked to be a parent column*,
  - table **pa-tab** is *marked to be a parent table*.

Note that child marking specifies parents, whereas parent marking does not specify children. Two tables are said to be in a *subordination relation via co-ide*, in symbols

ch-ide subordinated [co-ide] pa-ide

if both corresponding relations hold, i.e., if

- ch-ide content-subordinated [co-ide] pa-ide, and
- ch-ide marked-subordinated [co-ide] pa-ide.

A database is said to be *consistent* (OWN) if all tables marked as subordinated are content subordinated, although not necessarily vice-versa. Contents of two tables may “happen to be subordinated,” but if they are not marked as subordinated, the SQL engine does not need to protect their subordination. In the opposite case potential consistency violation must be prevented either by signaling an error and aborting program execution or by an appropriate modifications of tables. These mechanisms will be described in Sec. 11.3.4.

Since the types of tables in a database describe user-defined properties of the database, they are called *integrity constraints* of the database.

If a column marking of a column named **co-ide** includes a parent-table name **pa-ide** then the pair of identifiers (co-ide, pa-ide) is called a *coupling pair*.

The following operations on tables may violate database consistency:

1. when we modify a column marked as a parent:
  - a. a violation of repetition-freeness of the column; this situation may happen if we:
    - i. add a new row to the table, i.e., add a new value to the column,
    - ii. replace a value in the column by another value,
  - b. a violation of parent adequacy of the column; this situation may happen if we:
    - i. remove a row from the table, i.e., remove a value from the column,
    - ii. replace a value in the column by another value,
2. when we modify a column marked as a child:
  - a. a violation of child adequacy of the column; this situation may happen if we:
    - i. add such a row to the table that the value added to the column does not appear in the parent column,
    - ii. replace a value in the column by a value that does not appear in the parent.

In the sequel, all operations on tables will be defined in such a way that the consistency of databases will be secured either by aborting a consistency-braking action or by initiating a cascade of recovery actions. The following auxiliary function will be used to check if the subordination relation holds in a database between three identifiers. This function returns either a truth value *tv* or an error message that indicates why the relation does not hold.

subordination : DatBas x Identifier x Identifier x Identifier  $\mapsto$  {‘OK’} | Error

subordination.(dab, ch-ide, pa-ide, co-ide) =

|   |                                       |
|---|---------------------------------------|
| dab.ch-ide = ?                                | → ‘no child table in the base’        |
| dab.pa-ide = ?                                | → ‘no parent table in the base’       |
| <b>let</b>                                    | for i = 1;n                           |
| (ch-ctc, ch-tat) = dab.ch-ide                 |                                       |
| (ch-tah, ch-roy) = ch-tat                     |                                       |
| (pa-ctc, pa-tat) = dab.pa-ide                 |                                       |
| (pa-tah, pa-roy) = pa-tat                     |                                       |
| ch-tah.co-ide = ?                             | → ‘no such column in child’           |
| pa-tah.co-ide = ?                             | → ‘no such column in parent’          |
| <b>let</b>                                    |                                       |
| ch-col  | = ch-ctc.co-ide      child column     |
| (ch-bat, ch-bav, ch-coy, ch-com)              | = ch-tah.co-ide                       |
| pa-col  | = pa-ctc.co-ide      parent column    |
| (pa-bat, pa-bav, pa-coy, pa-com)              | = pa-tah.co-ide                       |
| ‘primary’ /: pa-com                           | → ‘parent not marked’                 |
| pa-ide /: ch-com                              | → ‘parent not expected’               |
| are-reperitions.pa-col                        | → ‘repetitions in parent column’      |
| elements.ch-col / $\subseteq$ elements.pa-col | → ‘missing elements in parent column’ |
| <b>true</b>                                   | → ‘OK’                                |

Of course

ch-ide subordination [co-ide] pa-ide holds in dba iff  
subordination.(dab, co-ide, ch-tab, pa-tab) = 'OK'.

One of typical cases when a database is not consistent is when some of its child-marked tables are orphans.

A column named **co-ide** is a *column orphan* of a parent table named **pa-ide** if it is marked as a child of **pa-ide** but content-wise it is not its child. A table is said to be a *table orphan* of another table if it includes a column orphan of that other table. Note that only children may be orphans.

The following function checks if a given column is an orphan in a given table of a given database. If that is the case, it signalizes this fact with 'YES' and describes the cause. In the opposite case it signalizes 'NOT'. It may also return an "unlabeled" error.

```
col-is-orphan : DatBas x Identifier x Identifier  $\mapsto$  {'NOT'} | {'YES'} x Error | Error
col-is-orphan.(dab, ta-ide, co-ide) =                                ta- table, co- column
  dab.ta-ide = ?                                 $\rightarrow$  'child table not in the database'
  let
    (ch-ctc, ch-tat) = dab.ta-ide                                child table, to be checked
    (ch-tah, ch-roy) = ch-tat                                    the type of this table
  ch-tah.co-ide = ?                                 $\rightarrow$  'no such column in child'
  let
    ch-col = ch-ctc.co-ide                                child column to be checked
    (ch-bat, ch-bav, ch-coy, ch-com) = ch-tah.co-ide          the type of the child column
    parents-co = ch-com - {'primary'}                        the set of names of parent columns
    parents-co = {}                                 $\rightarrow$  'NOT' co-ide is not an orphan, since it is not a child
  let
    {pa-ide-1, ..., pa-ide-n} = parents-co                                the names of parent tables
  dab.pa-ide-i = ?                                 $\rightarrow$  ('YES', 'parent not in the base') for i = 1;n
  let
    (pa-ctc-i, pa-tat-i) = dab.pa-ide-i for i = 1;n
  pa-ctc-i.co-ide = ?                                 $\rightarrow$  ('YES', 'column not in the parent') for i = 1;n
  let
    pa-col-i = pa-ctc-i.co-ide (parent column) for i = 1;n
    (pa-tah-i, pa-roy-i) = pa-tat-i (type of parent table) for i = 1;n
    (pa-bat-i, pa-bav-i, pa-coy-i, pa-com-i) = pa-tah-i.co-ide (type of parent column) for i = 1;n
    'primary' /: pa-com-i                                 $\rightarrow$  ('YES', 'parent column unmarked') for i = 1;n
    elements.ch-col  $\subseteq$  elements.pa-col-i                 $\rightarrow$  ('YES', 'incomplete parent column') for i = 1;n
    are-repetitions.pa-col-i                             $\rightarrow$  ('YES', 'repetitions in parent column') for i = 1;n
  true                                 $\rightarrow$  'NOT'
```

As we see, this check indicates an orphan in five following cases:

- (1) at least one parent-marked table does not exist in the database,
- (2) in at least one parent-marked tables, the parent column does not exist,
- (3) in at least one parent-marked table, the parent column is not marked with 'PRIMARY',
- (4) in at least one parent-marked table, its parent column does not include all expected elements,
- (5) at least one parent column is not repetition-free..

The following predicate checks if a table is an orphan of one of its parents:

```
tab-is-orphan : DatBas x Identifier  $\mapsto$  {'NOT'} | {'YES'} x Error | Error
tab-is-orphan.(dab, ta-ide) =
  dab.ta-ide = ?                                 $\rightarrow$  'no such table'
  let
    (ctc, tat) = dab.ta-ide
    {ide-1, ..., ide-n} = dom.ctc
  col-is-orphan.(dab, ta-ide, ide-i)  $\neq$  'NOT'  $\rightarrow$  col-is-orphan.(dab, ta-ide, ide-i) for i = 1;n
```

**true** $\rightarrow$  'NOT'

table is not an orphan

Subsequent predicate checks if for a column marked as a parent there are its children that are orphans:

 $\text{col-has-orphan} : \text{DatBas} \times \text{Identifier} \times \text{Identifier} \mapsto \{\text{'YES'}\} \mid \text{Error}$ 
 $\text{col-has-orphan}(\text{dab}, \text{pa-ide}, \text{co-ide}) =$ 
 $\text{dab.pa-ide} = ?$ 
 $\rightarrow$  'no such table'**let**
 $\{\text{ide-1}, \dots, \text{ide-n}\} = \text{dom.dab}$ 

the names of all tables in the current base

 $(\text{ctc-i}, (\text{tah-i}, \text{roy-i})) = \text{dab.ide-i} \quad \text{for } i = 1; n$ 

tables in the current base

 $(\text{ctc}, (\text{tah}, \text{roy})) = \text{dab.pa-ide}$ 

parent-marked table

 $\text{tah.co-ide} = ?$ 
 $\rightarrow$  'no such column'
 $\text{'primary'} /: \text{marking}(\text{tah.co-ide})$ 
 $\rightarrow$  'column is not a parent'
 $\text{are-repetitions}(\text{ctc.co-ide})$ 
 $\rightarrow$  'parent not repetition-free'
 $(\exists i) (\text{ide-i} : \text{marking}(\text{tah.co-ide}) \text{ and}$ 
 $\text{ctc-i.co-ide} = ! \quad \text{and}$ 
 $\text{elements}(\text{ctc-i.co-ide}) / \subseteq \text{elements}(\text{ctc.co-ide})) \rightarrow$  'orphan exists'
**true** $\rightarrow$  'YES'

no orphans

## 11.2.6 States

Intuitively, the enrichment of **Lingua** to **Lingua-SQL** consists in adding some new mechanisms to our language, whereas all former mechanisms are retained. There seem to be two alternative solutions to achieve this goal:

1. an enlargement of the domains of values, types and references by new elements whereas the structure of states remain unchanged,
2. an expansion of the structure of states by SQL components carrying storable SQL entities.

We chose the second solution since it seems more convenient to show, on the one hand, the borderline between the source language and its SQL part and, on the other — an interface between these two components. We assume, therefore, that our *new states* will be pairs:

 $\text{nes} : \text{NewSta} = \text{State} \times \text{SqlSta}$ 

consisting of a former state, which we shall call a *basic component*, and a *SQL component* that will carry tables, databases and transactions. We take the following introductory assumptions about our SQL model:

1. Time-and-date data, types and typed data will be included into **Lingua** domains of simple data, types and typed data respectively. Consequently variables pointing to these typed data will be stored in basic components of states.
2. SQL types and yokes will not be storable.
3. We do not introduce a covering relation between SQL types; types compatibility simply means that the involved types must be equal.
4. We shall not introduce expressions that generate tables and databases. Both will be created by declaration and then assigned to appropriate variables.
5. In SQL stores, tables and databases, will be assigned to variables directly, rather than via references. We do not introduce references in the SQL part of our model since objects will not carry neither tables nor databases.
6. Tables and databases will be public.

The domain of SQL states is defined as follows:

 $\text{sta} : \text{SqlSta} = \text{TraEnv} \times \text{SqlSto}$ 

SQL states

 $\text{tre} : \text{TraEnv} = \text{Identifier} \Rightarrow \text{Transaction}$ 

transaction environments

 $\text{sto} : \text{SqlSto} = \text{Database} \times \text{DbRep}$ 

SQL stores

 $\text{dbs} : \text{DbRep} = \text{Identifier} \Rightarrow \text{Database}$ 

database repository

Transactions are blocks of table-modifying instructions and will be described in Sec. 11.3.4.3.

The (unique) database that is a component of an SQL store is the *currently active database*, and its tables are called *active tables*. We assume that only one database may be active at a time. Database repository binds databases that are available but currently *not active*.

A typical new state is, therefore, of the following form:

((cle, tye, cov), (obn, dep, ota, sft, err)), (tre, (dab, dbr)) )

A new state is said to be *well-formed*, if:

1. its basic component is well-formed (Sec. 5.3),
2. every identifier appearing in a new state, appears in it only once,
3. the active database and all databases in the repository are consistent.

The set of well-formed new states will be denoted by

WfNewSta

Functions is-error, error, ◀ and declared (Sec. 5.3) are extended to new states in an obvious way.

## 11.3 The algebra of denotations

### 11.3.1 Replicated denotations and their constructors

To incorporate in our model the rule that all the mechanism of **Lingua** are retained in **Lingua-SQL**, we introduce the concept of a *replicated denotation*. Given a state-dependent denotation of **Lingua**, e.g., an imperative denotation

$\text{den} : \text{WfState} \rightarrow \text{WfState}$

by the *replica* of  $\text{den}$ , we mean the following function on new states:

$R[\text{den}] : \text{WfNewState} \rightarrow \text{WfNewState}$

$R[\text{den}].(\text{sta}, \text{sql-sta}) =$

$\text{den.sta} = ? \rightarrow ?$

**true**  $\rightarrow (\text{den.sta}, \text{sql-sta})$

A denotation is said to be a *replicated denotation* if it is a replica of a **Lingua** denotation. Replicated applicative denotations, i.e., the denotations of expressions, are defined in an analogous way.

Replicas of state-independent denotations, i.e., of primitive denotations (cf. Sec. 6.1) are just these denotations.

Now, with every domain  $\text{Den}$  of **Lingua** denotations we associate a corresponding domain of replicated denotations:

$R[\text{Den}] = \{R[\text{den}] \mid \text{den} : \text{Den}\}$

Carrier  $R[\text{Den}]$  is called a replica of the source carrier  $\text{Den}$ . Further on, with every **Lingua** constructor of denotations

$\text{cons} : \text{Den-1} \times \dots \times \text{Den-n} \mapsto \text{Den}$

we associate a *replica of this constructor* defined in the following way:

$R[\text{cons}] : R[\text{Den-1}] \times \dots \times R[\text{Den-n}] \mapsto R[\text{Den}]$

$R[\text{cons}].(R[\text{den-1}], \dots, R[\text{den-n}]) = R[\text{cons}].(\text{den-1}, \dots, \text{den-n})]$

Note that this definition is correct, since  $R$  is a 1-1 function. The following example illustrates the idea of replication:

$\text{while} : \text{ValExpDen} \times \text{InsDen} \mapsto \text{InsDen}$

$R[\text{while}] : R[\text{ValExpDen}] \times R[\text{InsDen}] \mapsto R[\text{InsDen}]$

$R[\text{while}].(R[\text{ved}], R[\text{ins}]) = R[\text{while}].(\text{ved}, \text{ind})]$

i.e.

$$\begin{aligned}
& R[\text{while}].(R[\text{ved}], R[\text{ins}]).(\text{sta}, \text{sql-sta}) = \\
& \quad \text{while} . (\text{ved}, \text{ins}).\text{sta} = ? \rightarrow ? \\
& \quad \text{true} \rightarrow (\text{while} . (\text{ved}, \text{ins}).\text{sta}, \text{sql-sta})
\end{aligned}$$

In Sec. 6.1, by **AlgDen** we have denoted the algebra of denotations of **Lingua**. Let **AlgDenSQL** denote our future algebra of **Lingua-SQL** denotations. We assume that all constructors of **AlgDen** will have their corresponding replicas in **AlgDenSQL** which implies that all reachable denotations of **AlgDen** will have their replicas in **AlgDenSQL**. The remaining denotations and their constructors will correspond to SQL mechanisms, and will be described in subsequent sections.

### 11.3.2 The carriers of the algebra of denotations

In our new algebra of denotations **AlgDenSQL** we are going to have three categories of carriers:

1. all primitive carriers of the former algebra plus two new primitive carriers of check settings and column markings,
2. *imitations* of carriers of state-dependent denotations of **AlgDen**; we assume that imitation carriers include replicas of corresponding **Lingua** denotations but may include also some new elements,
3. new SQL carriers.

Metavariables running over replicated domains will be denoted by the same symbols as for the corresponding original domains.

#### Primitive carriers

|     |              |                                     |                             |
|-----|--------------|-------------------------------------|-----------------------------|
| ide | : Identifier | = ...                               | identifiers                 |
| prs | : PriSta     | = {'private', 'public'}             | privacy statuses indicators |
| loi | : ListOfIde  | = Identifier <sup>c*</sup>          | lists of identifiers        |
| cli | : ClaInd     | = {'empty-class'}   Identifier      | class indicators            |
| chs | : CheSetting | = {'restrict', 'cascade'}           | check settings              |
| com | : ColMar     | = FinSub.(Identifier   {'primary'}) | column markings             |

#### Applicative carriers

|     |                 |                       |                                       |
|-----|-----------------|-----------------------|---------------------------------------|
| ted | : ImiTypeExpDen | = WfNewSta → TypeE    | imitated type-expression denotations  |
| ved | : ImiValExpDen  | = WfNewSta → ValueE   | imitated value-expression denotations |
| bvd | : BasValExpDen  | = WfNewSta → BasValE  | basic-value-expression denotations    |
| btd | : BasTypeExpDen | = WfNewSta → BasTypeE | basic-type expression denotations     |
| red | : RowExpDen     | = WfNewSta → LabRowE  | row expression denotations            |
| ryd | : RowYokExpDen  | = WfNewSta → RowYokE  | row-yoke expression denotations       |
| cyd | : ColYokExpDen  | = WfNewSta → ColYokE  | column-yoke expression denotations    |
| cmd | : ColMarExpDen  | = ColMar              | column-marking expression denotations |
| ctd | : ColTypeExpDen | = WfNewSta → ColType  | column-type expression denotations    |
| thd | : TabHeaExpDen  | = WfNewSta → TabHeaE  | table-header expression denotations   |
| ttd | : TabTypeExpDen | = WfNewSta → TabTypeE | table-type expression denotations     |

#### Imperative carriers

|     |                |                                    |                                       |
|-----|----------------|------------------------------------|---------------------------------------|
| dcd | : ImiDecDen    | = WfNewSta → WfNewSta              | imitated declaration denotations      |
| pod | : ImiProOpeDen | = WfNewSta ↦ WfNewSta              | imitated procedure opening denotation |
| ctd | : ImiClaTraDen | = Identifier ↦ WfNewSta → WfNewSta | imit. class-trans. denotations        |
| ind | : ImiInsDen    | = WfNewSta → WfNewSta              | imitated instruction denotations      |
| ppd | : ImiProPreDen | = WfNewSta → WfNewSta              | imitated program preamble denotations |
| prd | : ImiProDen    | = WfNewSta → WfNewSta              | imitated program denotations          |
| tdd | : TabDecDen    | = WfNewSta ↦ WfNewSta              | table-declaration denotation          |
| nid | : NewInsDen    | = WfNewSta → WfNewSta              | new instruction denotations           |

#### Declaration-oriented carriers



|     |                |                             |                                  |
|-----|----------------|-----------------------------|----------------------------------|
| dse | : ImiDecSec    | = ListOfIde x ImiTypeExpDen | new declaration sections         |
| fpd | : ImiForParDen | = ImiDecSec <sup>c*</sup>   | new formal-parameter-denotations |
| apd | : ActParDen    | = ListOfIde                 | new actual-parameter-denotations |

### Signature carriers

|     |                   |                                |                                       |
|-----|-------------------|--------------------------------|---------------------------------------|
| ips | : ImiImpProSigDen | = ImiForParDen x ImiForParDen  | new imperative-procedure sig. den.    |
| fps | : ImiFunProSigDen | = ImiForParDen x ImiTypeExpDen | new functional-procedure sig. den.    |
| ocs | : ImiObjConSigDen | = ImiForParDen x Identifier    | new object-constructor signature den. |

The constructors of our new algebra include replicas of all former constructors, plus some new constructors that we shall call *SQL constructors*. In the following sections we shall define a few most typical examples of SQL constructors.

## 11.3.1 Constructors of primitive denotations

The only new constructors in this category are constructors of column markings defined in Sec. 11.2.2.

## 11.3.2 Expressions

### 11.3.2.1 Categories of SQL expressions

Whereas in the case of **Lingua** we had only three categories of expressions — value expressions, type expressions and yoke expressions — in **Lingua-SQL** we add nine new categories:

1. basic-value expressions,
2. basic-type expressions,
3. row expressions,
4. row-yoke expressions,
5. column-yoke expressions,
6. column-marking expressions,
7. column-type expressions,
8. table-header expressions,
9. table-type expressions.

### 11.3.2.2 Basic-value expressions

Basic values are the only entities that can stand in table columns (cf. Sec. 11.2.2). They will be generated by basic-value expressions whose denotations belong to the domain:

$$\text{bvd} : \text{BasValExpDen} = \text{WfNewSta} \rightarrow \text{BasValE}$$

The introduction of this domain is an engineering decision. Note that an alternative might be to assume that basic-value expressions are included in the category of imitated value expressions. In that case, whenever a constructor of an SQL denotation has to generate a basic value, e.g., to put this value into an array, it has to (dynamically) check if the generated value is indeed basic. Although such solution is possible, a more optimal one seems to move this check to the level of syntax. In that case the job is made by a syntactic analyzer that checks if the used expression is a basic-value expression. To implement this solution, basic value expressions must constitute “an independent” carrier in the algebra of syntax, hence at the level of denotations we have to build a corresponding carrier as well.

In our model, we introduce two categories of value-expression:

1. expressions referring to basic-value operations,
2. expressions converting integers, reals and texts into corresponding basic values.

Since defining corresponding constructors of denotations is a fairly routine task, we shall skip it, the more that we have not defined constructors for basic data. We leave this issue open as a parameter of our model. We only assume to have a constructor that builds a denotation that generates an  $\Omega$  value of a given type:



$\text{bvd-generate-empty-value} : \text{BasTypeExpDen} \mapsto \text{BasValExpDen}$  i.e.  
 $\text{bvd-generate-empty-value} : \text{BasTypeExpDen} \mapsto \text{WfNewSta} \rightarrow \text{BasValE}$   
 $\text{bvd-generate-empty-value.btd.nes} =$   
 $\text{is-error.nes} \rightarrow \text{error.nes}$   
 $\text{btd.nes} : \text{Error} \rightarrow \text{btd.nes}$   
**let**  
 $\text{bat} = \text{btd.nes}$   
**true**  $\rightarrow (\Omega, \text{bat})$

### 11.3.2.3 Basic-type expressions

Although we have assumed (Sec. 11.2.6) that basic types are not storable (an engineering decision) their denotations must be functions on states, since decimal- and character types include integers, that we have to “somehow generate”. For this sake we shall use basic-value expressions.

$\text{btd} : \text{BasTypeExpDen} = \text{WfNewSta} \rightarrow \text{BasTypeE}$

An example of a constructor in this domain may be the following:

$\text{btd-create-decimal} : \text{BasValExpDen} \times \text{BasValExpDen} \mapsto \text{BasTypeExpDen}$  i.e.  
 $\text{btd-create-decimal} : \text{BasValExpDen} \times \text{BasValExpDen} \mapsto \text{WfNewSta} \rightarrow \text{BasTypeE}$   
 $\text{btd-create-decimal.bvd-1.bvd-2.nes} =$   
 $\text{is-error.nes} \rightarrow \text{error.nes}$   
 $\text{bvd-i.nes} = ? \rightarrow ?$  for  $i = 1, 2$   
**let**  
 $\text{val-i} = \text{bvd-i.nes}$  for  $i = 1, 2$   
 $\text{val-i} : \text{Error} \rightarrow \text{val-i}$  for  $i = 1, 2$   
 $\text{sort-t.val-i} \neq \text{'big-integer'} \rightarrow \text{'big-integer expected'}$  for  $i = 1, 2$   
 $\text{val-i} < 0 \text{ or } \text{val-i} > \text{max-i} \rightarrow \text{'parameter out of scope'}$  for  $i = 1, 2$   
**true**  $\rightarrow (\text{'De'}, \text{val-1}, \text{val-2})$

where  $\text{max-i}$ 's are parameter indicating maximal numbers in decimal types. The definitions of remaining constructors of this category are analogous.

It should be emphasized that formally basic types in SQL differ from the simple types defined in Sec. 4.2 and, therefore, simple type will not be used in building column types (Sec. 11.3.2.8). We have adopted this technical solution for the simplicity of our model.

### 11.3.2.4 Row expressions

The denotations of row expressions, given states return labelled rows:

$\text{red} : \text{RowExpDen} = \text{WfNewSta} \mapsto \text{LabRowE}$

The first constructor of these denotations creates a denotation that given a state creates a one-element row with empty field:

$\text{red-build-empty-field-row} : \text{Identifier} \mapsto \text{RowExpDen}$  i.e.  
 $\text{red-build-empty-field-row} : \text{Identifier} \mapsto \text{WfNewSta} \rightarrow \text{LabRowE}$   
 $\text{red-build-empty-field-row.ide.nes} =$   
 $\text{is-error.nes} \rightarrow \text{error.nes}$   
**true**  $\rightarrow [\text{ide}/\Omega]$

Next constructors builds a denotation that creates a one-element row data with a non-empty field:

$\text{red-build-nonempty-field-row} : \text{Identifier} \times \text{BasValExpDen} \mapsto \text{RowExpDen}$  i.e.  
 $\text{red-build-nonempty-field-row} : \text{Identifier} \times \text{BasValExpDen} \mapsto \text{WfNewSta} \rightarrow \text{LabRowE}$   
 $\text{red-build-nonempty-field-row.ide.bvd.nes} =$   
 $\text{is-error.nes} \rightarrow \text{error.nes}$   
 $\text{bvd.nes} = ? \rightarrow ?$

```

bvd.nes : Error      → bvd.nes
let
  val = bvd.nes
sort-t.val /: BasTyp → 'basic-type expected'
true              → [ide/val]

```

The third constructor corresponds to adding an empty field to a row data:

```

red-add-empty-field : Identifier x RowExpDen ↦ RowExpDen      i.e.
red-add-empty-field : Identifier x RowExpDen ↦ WfNewSta → LabRowE
red-add-empty-field.ide.red.nes =
  is-error.nes      → error.nes
  red.nes = ?       → ?
  red.nes : Error   → red.nes
let
  lar = red.nes
  red.ide = !       → 'identifier already bound'
true             → red[ide/Ω]

```

The last constructor corresponds to adding a nonempty field to a row data:

```

red-add-nonempty-field : Identifier x BasValExpDen x RowExpDen ↦ RowExpDen

```

We skip its obvious definition.

### 11.3.2.5 Column-yoke expressions

*Column-yoke expressions* generate column yokes and therefore their denotations constitute the following domain:

```

cyd : ColYokExpDen = WfNewSta → ColYokE

```

The constructors of column-yoke expression denotations are derived from corresponding yoke constructors. We give two examples to illustrate this idea.

```

cyd-qcy-greater-in : BasValExpDen ↦ ColYokExpDen      i.e.
cyd-qcy-greater-in : BasValExpDen ↦ WfNewSta → ColYokE
cyd-qcy-greater-in.bvd.nes =
  is-error.nes → error.nes
  bvd.nes = ?  → ?
let
  val = bvd.nes
  val : Error → val
true      → qcy-greater-in.val

```

This constructor builds a denotation that generates a quantified yoke that checks if all elements of a column are greater than a given integer. Next constructor builds a denotation that generates a holistic yoke that check if a column has repetitions

```

cyd-hcy-unique : ↦ ColYokExpDen
cyd-hcy-unique : ↦ WfNewSta → ColYokE
cyd-hcy-unique.().nes = hcy-unique.()

```

### 11.3.2.6 Row-yoke expressions

Analogously to the former expressions *row-yoke expressions* evaluate to row yokes:

```

ryd : RowYokExpDen = WfNewSta → RowYokE

```

The following constructor builds a denotation that generates a row yoke that adds two integers generated by two given row yokes:

```

ryd-add-int : RowYokExpDen x RowYokExpDen ↦ RowYokExpDen      i.e.

```

$\text{ryd-add-int} : \text{RowYokExpDen} \times \text{RowYokExpDen} \mapsto \text{WfNewSta} \rightarrow \text{RowYokE}$   
 $\text{ryd-add-int}(\text{ryd-1}, \text{ryd-2}).\text{nes} =$   
 $\text{is-error.nes} : \text{Error} \rightarrow \text{error.nes}$   
 $\text{ryd-i.nes} = ? \rightarrow ? \quad \text{for } i = 1,2$   
 $\text{ryd-i.nes} : \text{Error} \rightarrow \text{ryd-i.nes} \quad \text{for } i = 1,2$   
**let**  
 $\text{roy-i} = \text{ryd-i.nes} \quad \text{for } i = 1,2$   
**true**  $\rightarrow \text{ry-add-in}(\text{roy-1}, \text{roy-2})$

### 11.3.2.7 Column-marking expressions

Column-marking expressions are state-independent and therefore their denotations are just column markings

$\text{cmd} : \text{ColMarExpDen} = \text{ColMar}$

and their constructors are column-marking constructors defined in Sec. 11.2.2.

### 11.3.2.8 Column-type expressions

*Column-type expressions* evaluate to column types:

$\text{ctd} : \text{ColTypExpDen} = \text{WfNewSta} \rightarrow \text{ColTypE}$

We need only one constructor to build their denotations:

$\text{ctd-create} : \text{BasTypExpDen} \times \text{BasValExpDen} \times \text{ColYokExpDen} \times \text{ColMarExpDen} \mapsto \text{ColTypExpDen}$   
 $\text{ctd-create} : \text{BasTypExpDen} \times \text{BasValExpDen} \times \text{ColYokExpDen} \times \text{ColMarExpDen} \mapsto$   
 $\text{WfNewSta} \rightarrow \text{ColTypE}$

$\text{ctd-create}(\text{btd}, \text{bvd}, \text{cyd}, \text{cmd}).\text{nes} =$   
 $\text{is-error.nes} \rightarrow \text{error.nes}$   
 $\text{btd.nes} = ? \rightarrow ?$   
 $\text{bvd.nes} = ? \rightarrow ?$   
 $\text{cyd.nes} = ? \rightarrow ?$   
**let**  
 $\text{bat} = \text{btd.nes}$   
 $\text{val} = \text{bvd.nes}$   
 $\text{cyd} = \text{cyd.nes}$   
 $\text{bat} : \text{Error} \rightarrow \text{bat}$   
 $\text{val} : \text{Error} \rightarrow \text{val}$   
 $\text{cyd} : \text{Error} \rightarrow \text{cyd}$   
 $\text{sort-t.val} \neq \text{bat} \rightarrow \text{'inconsistent types'}$   
**true**  $\rightarrow (\text{bat}, \text{val}, \text{cyd}, \text{com})$

### 11.3.2.9 Table-header expressions

*Table-header expressions* evaluate to table headers (Sec. 11.2.4):

$\text{thd} : \text{TabHeaExpDen} = \text{WfNewSta} \rightarrow \text{TabHeaE}$

We need two constructors to build them

$\text{thd-create} : \text{Identifier} \times \text{ColTypExpDen} \mapsto \text{TabHeaExpDen}$   
 $\text{thd-add-column} : \text{Identifier} \times \text{ColTypExpDen} \times \text{TabHeaExpDen} \mapsto \text{TabHeaExpDen}$

We skip their obvious definitions.

### 11.3.2.10 Table-type expressions

*Table-type expressions* evaluate to table types:

$\text{ttd} : \text{TabTypExpDen} = \text{WfNewSta} \rightarrow \text{TabTypeE}$

They are built by one constructor:

$\text{ttd-create} : \text{TabHeaExpDen} \times \text{RowYokExpDen} \mapsto \text{TabTypExpDen}$

Its definition is obvious.

### 11.3.3 Declarations of table variables

A declaration of a table variable builds a one-row table with default values and assigns this table to an identifier in the active database. If the declared table expects parents then the declaration checks if such parents exist in the base. A consequence of this rule is that parents must be declared and filled with values prior to the declarations of their children (which makes sense biologically as well!).

|   |   |
|---|---|
| $\text{declare-table} : \text{Identifier} \times \text{TabTypExpDen} \mapsto \text{TabDecDen}$                            | i.e.  |
| $\text{declare-table} : \text{Identifier} \times \text{TabTypExpDen} \mapsto \text{WfNewSta} \rightarrow \text{WfNewSta}$ |   |
| $\text{declare-table}(\text{ta-ide}, \text{ttd}).\text{nes} =$  |   |
| $\text{is-error.nes}$   | $\rightarrow \text{nes}$  |
| $\text{declared.ta-ide.nes}$  | $\rightarrow \text{nes} \blacktriangleleft \text{'variable already declared'}$              |
| $\text{ttd.nes} = ?$  | $\rightarrow ?$   |
| $\text{ttd.nes} : \text{Error}$   | $\rightarrow \text{ttd.nes}$  |
| <b>let</b>  |   |
| $(\text{sta}, (\text{tre}, (\text{dab}, \text{dbr})))$  | $= \text{nes}$  |
| $\text{tat}$  | $= \text{ttd.nes}$  |
| $([\text{ide-1}/\text{cot-1}, \dots, \text{ide-n}/\text{cot-n}], \text{roy})$   | $= \text{tat}$  |
| $(\text{bat-i}, \text{bav-i}, \text{cyd-i}, \text{com-i})$  | $= \text{cot-i} \quad \text{for } i = 1;n$  |
| $\text{ctc}$  | $= [\text{ide-1}/(\text{bav-1}), \dots, \text{ide-n}/(\text{bav-n})]$                       |
| $\text{tab}$  | $= (\text{ctc}, \text{tat})$  |
| $\text{new-dab}$  | $= \text{dab}[\text{ta-ide}/\text{tab}]$  |
| $\text{row}$  | $= [\text{ide-1}/\text{bav-1}, \dots, \text{ide-n}/\text{bav-n}]$                           |
| $\text{cyd-i}(\text{bav-i}) : \text{Error}$   | $\rightarrow \text{cyd-i}(\text{bav-i})$  |
| $\text{cyd-i}(\text{bav-i}) = \text{fv}$  | $\rightarrow \text{'column yoke not satisfied'}$  |
| $\text{roy.row} : \text{Error}$   | $\rightarrow \text{roy.row}$  |
| $\text{roy.row} = \text{fv}$  | $\rightarrow \text{'row yoke not satisfied'}$   |
| $\text{tab-is-orphan}(\text{new-dab}, \text{ta-ide}) \neq \text{'NOT'}$   | $\rightarrow \text{nes} \blacktriangleleft \text{tab-is-orphan}(\text{dab}, \text{ta-ide})$ |
| <b>true</b>   | $\rightarrow (\text{sta}, (\text{tre}, (\text{new-dab}, \text{dbr})))$                      |

Note the difference between  $\text{ctc}$  which is a mapping from identifiers to one-element columns and  $\text{row}$  which is a mapping from identifiers to these elements.

The inspection of the definition of  $\text{col-is-orphan}(\text{dab}, \text{ta-ide}, \text{ide-i})$  in Sec. 11.2.5 shows that it generates an error only if  $\text{ta-ide}$  is not in  $\text{dab}$ , which in our case can't happen. Note also that the declared table is conformant with its type.

### 11.3.4 Instructions

#### 11.3.4.1 Row-oriented table instructions

The first constructor adds a row to a declared table. The new row is generated as a new labelled row by a row expression, and then every value in this row is added to a corresponding column. Two categories of type checks are necessary in this operation:

1. checking if the new table does not violate its type integrity, which may happen if:
  - a. an element added to a column is not of the expected type,
  - b. an element added to a column is  $\Omega$  whereas the default value of this column is different from  $\Omega$ ,
  - c. an element added to a columned barked as **PRIMARY** spoils the repetition freeness of this columns,
  - d. the new column violates the yoke of this column,
  - e. the added row violates the yoke row of the table;

- all these requirements will be checked by **type-table-check**,
- checking if the new table does not violate the database consistency, which may happen if we add a new element to a child column which makes the new column an orphan, i.e. if the added element does not belong to at least one parent columns of the child column;  
this requirement is checked by **col-is-orphan**.

### Add a row to a table

$\text{id-add-ro-to-ta} : \text{Identifier} \times \text{RowExpDen} \mapsto \text{NewInsDen}$  i.e.  
 $\text{id-add-ro-to-ta} : \text{Identifier} \times \text{RowExpDen} \mapsto \text{WfNewSta} \rightarrow \text{WfNewSta}$   
 $\text{id-add-ro-to-ta}(\text{ta-ide}, \text{red}).\text{nes} =$   
 $\text{is-error.nes} \rightarrow \text{nes}$   
**let**  
 $(\text{sta}, (\text{tre}, (\text{dab}, \text{dbr}))) = \text{nes}$   
 $\text{dab.ta-ide} = ? \rightarrow \text{'no such table'}$   
 $\text{red.nes} = ? \rightarrow ?$   
 $\text{red.nes} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{red.nes}$   
**let**  
 $\text{tab} = \text{dab.ta-ide}$   
 $(\text{ctc}, \text{tat}) = \text{tab}$   
 $[\text{ide-1/col-1}, \dots, \text{ide-n/col-n}] = \text{ctc}$  table-content to be modified  
 $\text{lar} = \text{red.nes}$  labeled row to be added  
 $\text{dom.lar} \neq \{\text{ide-1}, \dots, \text{ide-n}\} \rightarrow \text{'incorrect row identifiers'}$   
**let**  
 $\text{new-val-i} = \text{lar.ide-i}$  for  $i = 1;n$   
 $\text{new-col-i} = \text{col-i} @ (\text{new-val-i})$  for  $i = 1;n$   
 $\text{new-ctc} = [\text{ide-1/new-col-1}, \dots, \text{ide-n/new-col-n}]$  new column table-content  
 $\text{tab-message} = \text{type-table-check}(\text{new-ctc}, \text{tat})$  (Sec. 11.2.4)  
 $\text{tab-message} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{tab-message}$   
**let**  
 $\text{new-tab} = (\text{new-ctc}, \text{tat})$   
 $\text{new-dab} = \text{dab}[\text{ta-ide/new-tab}],$   
 $\text{dab-message-i} = \text{col-is-orphan}(\text{new-dab}, \text{ta-ide}, \text{ide-i})$   
 $\text{dab-message-i} \neq \text{'NOT'} \rightarrow \text{nes} \blacktriangleleft \text{dab-message-i}$  for  $i=1;n$   
**true**  $\rightarrow (\text{sta}, (\text{tre}, (\text{new-dab}, \text{dbr})))$

Note that if at any stage of our table modification an error message is raised, then it is loaded to the error register of the initial state, which means that the initial database remains unchanged and the intended modification is abandoned.

All constructors that follow will be defined according to the following scheme:

- the transformation of a column content of a table into a row content,
- a modification of the row content by an appropriate constructor of labelled rows,
- the transformation of the resulting row content into a new column content,
- checking integrity constraints of the new table,
- checking integrity constraints of the new database and possibly initiating a cascade action.

### Remove from a table all rows that satisfy a given yoke

$\text{id-cut-ro-from-ta} : \text{RowYokExpDen} \times \text{Identifier} \times \text{CheSetting} \mapsto \text{NewInsDen}$  i.e.  
 $\text{id-cut-ro-from-ta} : \text{RowYokExpDen} \times \text{Identifier} \times \text{CheSetting} \mapsto \text{WfNewSta} \rightarrow \text{WfNewSta}$   
 $\text{id-cut-ro-from-ta}(\text{ryd}, \text{ta-ide}, \text{chs}) =$   
 $\text{is-error.nes} \rightarrow \text{nes}$   
**let**  
 $(\text{sta}, (\text{tre}, (\text{dab}, \text{dbr}))) = \text{nes}$   
 $\text{dab.ta-ide} = ? \rightarrow \text{'no such table'}$

```

ryd.nes = ?                                → ?
ryd.nes : Error                            → ryd.nes
let
  roy          = ryd.nes
  (ctc, tat)    = dab.ide
  (lar-1,...,lar-k) = C2R.ctc
k = 1                                         → nes ◀ 'single row can't be removed'
let
  rtc          = drop-rows.(roy, (lar-1,...,lar-k))      row-table-content of the new table
  new-ctc       = R2C.rtc                                new column table-content
  tab-message   = type-table-check.(new-ctc, tat)
tab-message : Error                         → nes ◀ tab-message
let
  new-tab       = (new-ctc, tat)
  new-dab       = dab[ta-ide/new-tab]
  [ide-1/col-1,...,ide-n/col-n] = new-ctc
  (tah, roy)    = tat
  [ide-1/cot-1,...,ide-n/cot-n] = tah
  (bat-i, bav-i, coy-i, com-i) = cot-i    for i = 1;n
each of new parent columns may have orphans
col-has-orphan.(new-dab, ta-ide, ide-i) ≠ 'NOT' and                                for i = 1;n
col-has-orphan.(new-dab, ta-ide, ide-i) ≠ 'column is not a parent' →
  chs = 'restrict'      → nes ◀ col-has-orphan.(new-dab, ta-ide, ide-i)
  chs = 'cascade'      → ... cascading removal of orphans in the table
true                    → (sta, (tre, (new-dab, dbr)))

```

This constructor first transforms the given column content of the table into a row content. From this content it removes all labelled rows that satisfy the yoke. This action is performed by using an auxiliary function `drop-rows`; we skip its formal definition. The resulting row content of the table is transformed (back) into a column table-content. After this modification of our table we have to perform two checks.

In the first place we check if the new table content is of the former table type. Note that although the dropping of column elements won't spoil quantified yokes it may spoil holistic ones (Sec. 11.2.2). If this check is negative we simply generate an error message and the table remains unchanged.

In the second checking step, we check if removing some elements from parent-marked columns of our table did not cause them to have orphans. To do that we use function `col-has-orphan` and we check column-by-column the new table. Once we find a column which is a parent and has orphans, we either abort our instruction and report an error message—that in this case will be 'orphan exists'—or we initiate a cascade action of removing orphans, and their orphans, and their orphans...

E.g. (cf. Fig. 11.2-1), if the `Distribution` row is removed from `Affiliations`, then the column `Department_ID` in `Employees` becomes an orphan. If this happens, the further action depends on the check-setting `chs`.

- If it is set to 'restrict', then the operation is abandoned and an error is signaled.
- If it is set to 'cascade', then we remove from `Employees` all rows that include `Distribution`.

We shall not formalize the "cascade" part of our definition, as it is a technical task that would not significantly contribute to our model. Besides, none of the manuals known to us describe this option precisely enough to allow for its formalization. In particular, none even mentions what may happen if we have a circular reference between tables.

#### 11.3.4.2 Column-oriented table instructions

The four constructors to be defined in this section are associated with columns only implicitly since none of them neither takes columns as arguments nor return them as a results. All of them, except adding a column, are defined in five steps analogous to these of row-oriented constructors:

1. the transformation of a column table-content into a row table-content,
2. a modification of every labeled row by an appropriate constructor of labeled rows,
3. the transformation of the resulting row-table-content into a new table (in the cases of constructors **add**, **cut**, **change**) or into a column (in the case of constructor **get**).
4. checking integrity constraints of the new table,
5. checking integrity constraints of the new database.

### Add a column to a table

nid-add-column : Identifier x Identifier x ColTypExpDen  $\mapsto$  NewInsDen

nid-add-column : Identifier x Identifier x ColTypExpDen  $\mapsto$  WfNewSta  $\rightarrow$  WfNewSta

nid-add-column.(ta-ide, co-ide, ctd).nes = ta – table, co- column

is-error.nes  $\rightarrow$  nes

ctd.nes = ?  $\rightarrow$  ?

ctd.nes : Error  $\rightarrow$  ctd.nes

**let**

cot = ctd.nes

(sta, (tre, (dab, dbr)) ) = nes

dab.ta-ide = ?  $\rightarrow$  nes  $\blacktriangleleft$  'no such table'

**let**

(ctc, tat) = dab.ta-ide

table to be modified

ctc.co-ide = !  $\rightarrow$  nes  $\blacktriangleleft$  'column already exists'

**let**

n = depth.ctc

(bat, bav, coy, com) = cot

col = bav<sup>cn</sup>

column of default values of length n

col-message = check-column-type.(col, cot)

col-message  $\neq$  'OK'  $\rightarrow$  nes  $\blacktriangleleft$  col-message

**let**

new-ctc = ctc[co-ide/col]

new column table-content

(tah, roy) = tat

source table type

new-hea = tah[co-ide/cot]

new table header

new-tat = (new-hea, roy)

new table type

new-tab = (new-ctc, new-tat)

new table

new-dab = dab[ta-ide/new-tab]

new database

*here we start the consistency check of the new database*

com = {}  $\rightarrow$  (sta, (tre, (new-dab, dbr)))

'primary' : com  $\rightarrow$  nes  $\blacktriangleleft$  'no empty cells in a parent column'

**let**

{ide-1,...,ide-k} = com

result-i = subordination.(new-dab, ta-ide, ide-i, co-ide) for i = 1;k

result-i : Error  $\rightarrow$  nes  $\blacktriangleleft$  result-i for i = 1;k

**true**  $\rightarrow$  (sta, (tre, (new-dab, dbr)))

This constructor first builds a new column of (identical) default values indicated by the type of this column, and then it checks if the new column satisfies the assumed column type. If that is the case, the table type is augmented by the new column type but its row yoke remains unchanged. Note that due to the monotonicity of row yokes (cf. Sec. 11.2.3), adding a new column to a table does not violate its row yoke. If we want to cover new column by the row yoke, we have to modify the latter by a dedicated instruction.

In the end we perform the consistency check:

- if the new column is neither a marked parent not a marked child, then we are done,
- otherwise, if the column is marked as a parent, then we signalize an error, since our column includes repetitions; all its elements are equal to the default value,



- otherwise we identify all tables indicated as parents of our column (and table), and we check if they are actually parents.

### Cut a column from a table

The simple removal of a column from a table is an easy task, and therefore the bulk of the work is in checking if the resulting database satisfies all integrity constraints. There are essentially two cases when the removal of a column may destroy the integrity of a base:

1. A removal of a column from a table means a removal of an element from each row, and this may cause the row yoke of the table to be not satisfied anymore.
2. If the column to be removed is marked as a parent, then its removal may cause orphanhood of some tables in the base. In this case the actions to be taken depends on the setting the checking parameter:
  - a. if it is set to 'restrict', then an error message is generated and further action is aborted,
  - b. if it is set to 'cascade', then the instruction removes all these children columns from children tables that violate database consistency; during this action it permanently check if the row yokes of the modified tables are satisfied, and if they are not, it aborts the cascade and returns to the initial state.

nid-cut-column : Identifier x Identifier x CheSetting  $\mapsto$  NewInsDen

nid-cut-column : Identifier x Identifier x CheSetting  $\mapsto$  WfNewSta  $\rightarrow$  WfNewSta

nid-cut-column.(ta-ide, co-ide, chs).nes = ta – table, co- column

is-error.nes  $\rightarrow$  nes

**let**

(sta, (tre, (dab, dbr)) ) = nes

dab.ta-ide = ?  $\rightarrow$  nes  $\blacktriangleleft$  'no such table'

**let**

(ctc, tat) = dab.ta-ide

ctc.co-ide = ?  $\rightarrow$  nes  $\blacktriangleleft$  'no such column'

**let**

(tah, roy) = tat source table type

(bat, bav, coy, com) = tah.co-ide

new-ctc = ctc[co-ide/?] new column table-content

(row-1,...,row-n) = C2R.new-ctc

roy.row-i : Error  $\rightarrow$  nes  $\blacktriangleleft$  row-message-i for i = 1;n

roy.row-i = fv  $\rightarrow$  nes  $\blacktriangleleft$  'row yoke not satisfied' for i = 1;n

**let**

new-hea = tah[co-ide/?] new table header

new-tat = (new-hea, roy) new table type

new-tab = (new-ctc, new-tat) new table

'primary' /: com  $\rightarrow$  (sta, (tre, (dab[ta-ide/new-tab], dbr)) )

chs = 'restrict'  $\rightarrow$  nes  $\blacktriangleleft$  'a parent column can't be removed'

chs = 'cascade'  $\rightarrow$  ... conditional removal of all children that cause inconsistencies

### Filter the indicated columns of a table (remove the not-indicated)

va-filter-col-from-ta : ListOfIde x Identifier x CheSetting  $\mapsto$  NewInsDen

va-filter-col-from-ta : ListOfIde x Identifier x CheSetting  $\mapsto$  WfNewSta  $\rightarrow$  WfNewSta

In this definition, we refer to the domain of lists of identifiers ListOfIde (Sec. 6.1). We skip its formal definition.

### Modify a column in a table conditionally

The constructed denotation modifies all these elements of an indicated column that belong to rows satisfying a given row yoke. The new values in the indicated column are calculated by means of a modifying row yoke. An example of such an instruction may be the following:

```
UPDATE Employees
SET Salary      = Salary * 1,1
WHERE Position  = 'salesman'
```

The denotation of such instruction builds a new table in five steps:

1. it transforms the source column table-content into a row table-content,
2. it calculates the modified values of the target column,
3. it conditionally replaces in all rows the former values in the indicated columns by new values,
4. it checks if the new table content is compatible with the table's (unchanged) type,
5. it checks the consistency of the new database.

$\text{nid-modify-col-in-ta} : \text{Identifier} \times \text{Identifier} \times \text{RowYokExpDen} \times \text{RowYokExpDen} \mapsto \text{NewInsDen}$

$\text{nid-modify-col-in-ta} : \text{Identifier} \times \text{Identifier} \times \text{RowYokExpDen} \times \text{RowYokExpDen} \mapsto$

$\text{WfNewSta} \mapsto \text{WfNewSta}$

$\text{nid-modify-col-in-ta} . (\text{ta-ide}, \text{co-ide}, \text{mo-ryd}, \text{se-ryd}) =$

$\text{mo- modifying, se- selecting}$

$\text{is-error.nes} \rightarrow \text{nes}$

**let**

$(\text{sta}, (\text{tre}, (\text{dba}, \text{dbr}))) = \text{nes}$

$\text{dab.ta-ide} = ? \rightarrow \text{nes} \blacktriangleleft \text{'no such table'}$

**let**

$(\text{ctc}, \text{tat}) = \text{dab.ta-ide}$

$(\text{tah}, \text{roy}) = \text{tat}$

$\text{com} = \text{marking} . (\text{tah.co-ide})$

$\text{ctc.co-ide} = ? \rightarrow \text{nes} \blacktriangleleft \text{'no such column'}$

**let**

$(\text{lar-1}, \dots, \text{lar-k}) = \text{C2R.ctc}$

$\text{mo-bav-i} = \text{mo-roy.lar-i} \quad \text{for } i = 1;k$

$\text{se-bav-i} = \text{se-roy.lar-i}$

modified values

selection values

$\text{mo-bav-i} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{mo-bav-i} \quad \text{for } i = 1;k$

$\text{se-bav-i} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{se-bav-i} \quad \text{for } i = 1;k$

$\text{type.se-bav-i} \neq \text{'boolean'} \rightarrow \text{'boolean value expected'}$

**let**

$\text{new-lar-i} =$

$\text{se-bav-i} = \text{tv} \rightarrow \text{lar-i}[\text{co-ide}/\text{mo-bav-i}] \quad \text{for } i = 1;k$

**true**  $\rightarrow \text{lar-i}$

$\text{new-rtc} = (\text{new-lar-1}, \dots, \text{new-lar-k})$

$\text{new-ctc} = \text{R2C.new-rtc}$

$\text{tab-message} = \text{type-table-check} . (\text{new-ctc}, \text{tat})$

$\text{tab-message} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{tab-message}$

**let**

$\text{new-tab} = (\text{new-ctc}, \text{tat})$

$\text{new-col} = \text{new-ctc.co-ide}$

*here we start the consistency check of the database*

$\text{com} = \{\} \rightarrow (\text{sta}, (\text{tre}, (\text{dab}[\text{ta-ide}/\text{new-tab}], \text{dbr})))$

**let**

$\{\text{pa-ide-1}, \dots, \text{pa-ide-k}\} = \text{com} - \{\text{'primary'}\}$

parents declared

$\text{result-i} = \text{subordination} . (\text{dab}, \text{ta-ide}, \text{pa-ide-i}, \text{co-ide}) \quad \text{for } i = 1;k$

$\text{result-i} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{result-i} \quad \text{for } i = 1;k$

$\text{'primary'} : \text{com} \rightarrow$

$\text{are-repetitions.new-col} \rightarrow \text{'repetitions in a parent column'}$

**let**

$\text{orp-message} = \text{col-has-orphan} . (\text{dab}, \text{ta-ide}, \text{co-ide})$

$\text{orp-message} : \text{Error} \rightarrow \text{nes} \blacktriangleleft \text{orp-message}$

$\text{orp-message} = \text{tv} \rightarrow \text{nes} \blacktriangleleft \text{'orphan detected'}$

**true**  $\rightarrow (\text{sta}, (\text{tre}, (\text{dab}[\text{ta-ide}/\text{new-tab}], \text{dbr})))$

**true**  $\rightarrow (\text{sta}, (\text{tre}, (\text{dab}[\text{ta-ide}/\text{new-tab}], \text{dbr})))$

The consistency of a database may be destroyed by our instruction if the marking of the modified column is not empty and:

1. the modified column is marked as a parent and,
  - a. it is not repetition free, or
  - b. has orphans in the base,
2. the modified column is marked as a child but its table does not satisfy the expected subordination relations.

Note that if the selecting row yoke *se-ryd* identifies exactly one row, then our instruction may be also used to replace a current value in a table field by a new value. The table field is in such a case identified by a column name together with a row yoke that identifies the row where the field belongs.

#### 11.3.4.3 Transactions

As was announced in Sec. 10.6 *transactions* in SQL are blocs of sequentially composed instructions, i.e., composed instructions, enriched with special instructions protecting databases from the destruction of their integrity. In such cases a programmer in SQL may use a *security instruction* that orders the system to make a copy of the current database which the system may bring back if an operation can't be completed without errors.

It appears that in our model, the concept of protecting a database against integrity violations is partly incorporated into our instructions, where such violations are signaled by error messages and the state remains unchanged which corresponds to a **ROLLBACK** instruction. Since the formalization of transactional instructions listed in Sec. 10.6 seems obvious, we skip them.

The property of transactions called *isolation* can't be formalized in our model, since it does not include concurrency mechanisms. A denotational model of concurrency is discussed in Sec. 12, but we are not talking about SQL there.

#### 11.3.4.4 Queries

A query is an instructions that first creates a new table from another table or tables, and then displays it on the monitor. We skip formal definitions of their constructors.

#### 11.3.4.5 Instructions modifying integrity constraints

There are basically three categories of integrity-constraint modifications:

1. the modifications of an indicated column yoke of an indicated table,
2. the modification of the row yoke of an indicated table,
3. the modification of the current subordination marks.

The corresponding constructors may be easily defined in our model.

#### 11.3.4.6 Views

Views are virtually procedures that call table instructions. They may be introduced in our model in a similar way as in **Lingua**.

## 12 AN EXERCISE WITH A DENOTATIONAL CONCURRENCY

### 12.1 An overview of our model of concurrency

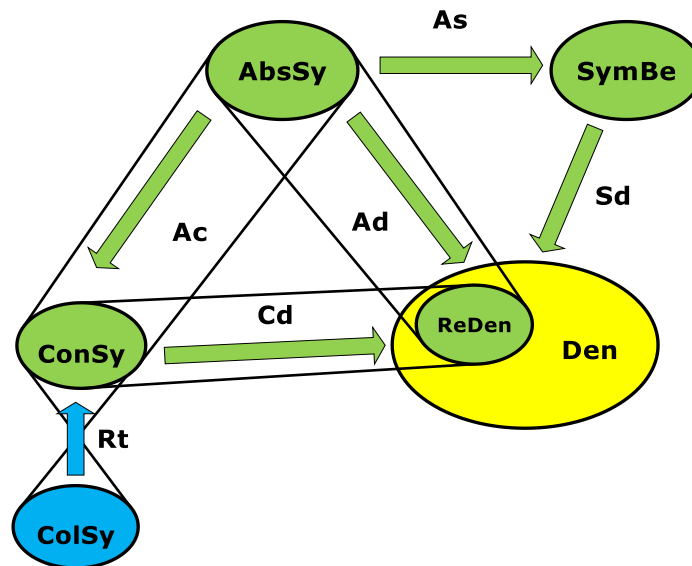
So far, our programs may be seen as flowchart-like structure where nodes are state-to-state functions. Since we have resigned from **goto**'s, our flowcharts are built by three constructors: sequential composition, branching, and looping. Note that when passing from **Lingua** to **Lingua-SQL**, we only enriched the variety of nodes of our flowcharts, but we preserved flowchart control structures and state-to-state denotations.

To incorporate a concurrency mechanisms into our model we shall go beyond both these paradigms:

- flowcharts will be replaced by Petri nets,
- state-to-state functions, i.e., sets of pairs of states, by sets of finite or infinite sequences of states called *bundles of computations*.

In both these cases we have to do with natural generalizations: Petri nets are generalizations of flowcharts, and sequence — of ordered pairs.

Independently of these modifications we shall also introduce a new *algebra of symbolic behaviors* between the algebra of abstract syntax and that of denotations (Fig. 12.1-1). Before we proceed to building our new model we shall roughly explain the role of the new algebra on the example of sequential programs.



**Fig. 12.1-1** An extended algebraic model of a programming language

The elements of the new algebra are called *symbolic behaviors*, and are sets of *symbolic executions*. The latter, in turn, are finite sequences of *atoms*, and atoms are the following program components:

- single declarations of all categories,
- single instructions that include:
  - assignments,
  - procedure calls,
  - value expressions

In which way value expressions are regarded as instructions, will be seen in a moment.

Now, with every program we assign its symbolic behavior, i.e., the set of its potential symbolic executions. From the view point of formal-language theory, behaviors may be regarded as regular languages over an alphabet of atoms. The step from symbolic behaviors to denotations consists in replacing:

- atoms by their denotations,
- sequences of atoms by sequential compositions of these denotations,
- sets of sequences of atoms by set-theoretic unions of the denotations of sequences; here a nondeterminism may come into play.

To describe our new model in a more formal way — still without going into technical details — let us show how it works in the case of instructions. In that case the corresponding domain of symbolic behaviors is defined as follows:

|     |             |  |                                     |
|-----|-------------|--|-------------------------------------|
| sbi | : SymBehIns | = Set.SymExeIns                        | symbolic behaviors of instructions  |
| sei | : SymExeIns | = InsAto <sup>c*</sup>                 | symbolic executions of instructions |
| ati | : Atolns    | = AsgIns   ProCal   ValExp             | atoms of instructions               |
| ain | : AsgIns    | = <b>assign</b> (AbsRefExp, AbsValExp) | assignment instructions             |
| ... |             |  |                                     |

Having these domains, we define a *symbolic semantics of instructions* as a function that assigns symbolic behaviors to (abstract-syntax) instructions:

$SSI : \text{AbsIns} \mapsto \text{SymBehIns}$

|   |   |
|---|---|
| $SSI[ati]$                                  | $= \{(ati)\}$   |
| $SSI[sin1 ; sin2]$                          | $= SSI[sin1] \odot SSI[sin2]$   |
| $SSI[\text{if exp then ins1 else ins2 fi}]$ | $= (\{(exp)\} \odot SSI[ins1]) \mid (\{(\text{not exp})\} \odot SSI[ins2])$ |
| $SSI[\text{while exp do ins od}]$           | $= (\{(exp)\} \odot SSI[ins1])^{c*} \odot \{(\text{not exp})\}$             |

where  $\odot$  is concatenation of languages (Sec. 2.5).

Once we are done with the mechanism of symbolic behaviors, we can pass to the level of denotations. We define a function from symbolic behaviors to state-to-state denotations:

$S2D : \text{SymBehIns} \mapsto \text{InsDen}$

First, with every (atomic) behavior that includes only one execution with only one atom we assign the (earlier defined) state-to-state denotations of this atom. E.g., (cf. Sec. 7.1)

$S2D[\{(\text{assign}(\text{AbsRefExp}, \text{AbsValExp}))\}] = A2D[\text{assign}(\text{AbsRefExp}, \text{AbsValExp})]$

Next, with every (abstract) value expression **ave** we assign the following state-to-state function that we shall call a *filter*:

$S2D[\{(ave)\}] : \text{State} \rightarrow \text{State}$   
 $S2D[\{(ave)\}].sta =$   
 $A2D[ave].sta = ? \rightarrow ?$   
**let**  
 $val = A2D[ave].sta$   
 $val : \text{Error} \rightarrow sta \leftarrow val$   
 $val /: \text{Bool} \rightarrow sta \leftarrow \text{'boolean expected'}$   
 $val = tv \rightarrow sta$   
 $val = fv \rightarrow ?$

The denotation of  $\{(ave)\}$  is a subset of the identity function on states. It is transparent for states evaluating **ave** to **tv** and otherwise generates an error or is undefined accordingly. The role of subsets of identity in the semantics of programs is explained in Sec. 8.3.

Finally, with every symbolic execution of an instruction we assign the (functional) composition of all the denotations of its elements, and with every symbolic behavior — the (set theoretic) union of the denotations of its executions:

$$\begin{aligned} \text{S2D}\{(\text{ati-1}, \dots, \text{ati-n}) \mid (\text{ati-i}, \dots, \text{ati-n}) : \text{sbi}\} = & \text{for all } \text{sbi} : \text{SymBehIns} \\ & \cup \{(\text{S2D}[\text{ati-1}] \bullet \dots \bullet \text{S2D}[\text{ati-n}] \mid (\text{ati-i}, \dots, \text{ati-n}) : \text{sbi}\} \end{aligned}$$

On the ground of our extended model of sequential languages the step from sequentiality to concurrency consists in:

- we replace symbolic behaviors, i.e., regular languages over an alphabet of atoms generated by flowcharts by *trace languages* of Antony Mazurkiewicz (see [77]) generated by simple Petri nets,
- we replace state-to-state denotations by bundles of computations of Andrzej Blikle (see [21], [22] and [24]).

Similarly as in the case of SQL, we shall limit our investigations to the denotational algebra of the new model, and we shall only sketch the construction rather than go into “practical technicalities”.

## 12.2 Bundles of computations

### 12.2.1 Abstract nets and quasinet

In Sec. 2.5 and Sec. 2.7 we have described a CPO of formal languages and of binary relations respectively. Both provide an adequate context for equational (fixed-point) descriptions of the syntaxes and the denotations of programming languages. Both are equipped with a continuous monoid operation: the concatenation of languages and the sequential composition of relations respectively. In our approach to concurrency we shall use yet another CPO with a monoid operation: a CPO of bundles of computations. All three are particular cases of abstract *nets* and *quasinet*s investigated by A.Blikle in the decade of 1970. in several papers (cf. [19], [21], [22], [23] and [24]). In this section we give a short introduction to their theory.

A *quasinet* is a partially ordered sets with a monoidal operation. By a *monoid* over a set  $A$  we mean a triple  $(A, \bullet, e)$

where  $\bullet$  is a total function, called *composition*

$$\bullet : A \times A \mapsto A$$

with two following properties

1.  $(a \bullet b) \bullet c = a \bullet (b \bullet c)$  — *associativity*
2.  $a \bullet e = e \bullet a = a$  —  $e$  is the *unit* of composition

By a *quasinet* we mean a quintuple  $(A, \sqsubseteq, \bullet, \Phi, e)$  with the following properties<sup>103</sup>:

- (1)  $(A, \sqsubseteq, \Phi)$  is a CPO
- (2)  $(A, \bullet, e)$  is a monoid
- (3)  $\bullet$  is continuous
- (4)  $\Phi \bullet a = \Phi$  for any  $a : A$

A *quasinet* is said to be a *net* if

- (5)  $a \bullet \Phi = \Phi$  for any  $a : A$ .

A *quasinet* is said to be *set-theoretic* if  $A$  is a set of sets and  $\sqsubseteq$  is an inclusion of sets. If there exist elements  $a$  and  $b$  both different from  $\Phi$  such that

$$a \bullet b = \Phi$$

then they are called the *divisors of zero*.

<sup>103</sup> Nets were originally introduced by A.Blikle in [19] as complete lattices with a monoidal operation. Here we are weakening our definition assuming that a net must be a CPO, since in fact all what we need is that continuous fixed-point equations have least solutions. Besides, set-theoretic CPO's of functions are not lattices since a union of two functions needs not to be a function.

Originally quasinetns were defined as complete lattices, rather than CPO's. A *complete lattice* is a POS  $(A, \sqsubseteq)$  where for every subset of  $A$  there exists the least upper bound and the greatest lower bound. Here we introduce nets and quasinetns understood as POS's in order to cover POS's of partial functions which do not constitute lattices.

## 12.2.2 Nets of the bundles of computations

In the majority of denotational models of programming languages, the denotations of programs are state-to-state functions, or — in non-deterministic cases — binary relations on states. This approach is adequate to situations where programs are supposed to terminate after a finite number of steps. If, however, we intend to deal with programs that “run forever” and we still regard them correct — as it is frequently the case with concurrent programs — binary relations become inadequate. In the first place on their ground we can't describe a situation where a nondeterministic program starting with a given input state may generate a terminating execution or alternatively an infinite execution. Besides, on the ground of relational semantics we can hardly talk about temporal properties of executions.

To tackle the mentioned problems we shall use a model where the denotations of programs are sets of sequences of states generated by programs. Such models were investigated by A. Blikle in [21], [22] and [24].

Let **State** be an arbitrary set of items called *states*. By a *computation* over **State** we mean any empty, finite, or infinite sequence of states. Let

|                                  |  |  |
|----------------------------------|--|--|
| $\text{com} : \text{Co.State}$   | $= \text{FiCo.State} \mid \text{InCo.State}$ | the set of all computations over <b>State</b>  |
| $\text{com} : \text{FiCo.State}$ | $= \text{State}^{c^*}$                       | the set of all finite computations over <b>State</b>   |
| $\text{com} : \text{InCo.State}$ | $= \text{State}^{c^\infty}$                  | the set of all infinite computations over <b>State</b><br>including the empty computation $()$ |

Consider two computations  $(\text{sta-11}, \dots, \text{sta-1n})$  and  $(\text{sta-21}, \dots, \text{sta-2m})$  with  $n, m \leq \infty$ , each of which may be empty ( $n=0$  or  $m=0$ ), finite ( $1 \leq n \leq k$  or  $1 \leq m \leq k$ ) or infinite ( $n=\infty$  or  $m=\infty$ ). By a *sequential composition* or just a *composition* of these computations we mean a computation defined in the following way:

|   |  |
|---|--|
| $(\text{sta-11}, \dots, \text{sta-1n}) \bullet (\text{sta-21}, \dots, \text{sta-2m}) =$ |  |
| $n = \infty$  | $\rightarrow (\text{sta-11}, \dots, \text{sta-1n})$                                      |
| $n = 0$   | $\rightarrow ()$   |
| $m = 0$   | $\rightarrow ()$   |
| $\text{sta-1n} \neq \text{sta-21}$  | $\rightarrow ()$   |
| <b>true</b>   | $\rightarrow (\text{sta-11}, \dots, \text{sta-1n}, \text{sta-22}, \dots, \text{sta-2m})$ |

Intuitively a sequential composition of two computations is a computation that starts with the first one, and continuous with the second. In turn, an empty computation  $()$  is a computation that “cannot happen”. With this interpretation an intuitive explanation of our definition is the following:

- an infinite computation followed by any other computation (even empty), is the former computation, because nothing may be added “at the end” of an infinite computation,
- a composition of a computation that can't happen with any computation is a computation that can't happen,
- a sequential composition of a finite computation with a computation that can't happen, cannot happen,
- a sequential composition of a finite computation whose last state is different from the first state of the second computation, i.e., where  $\text{sta-1n} \neq \text{sta-21}$ , can't happen because the second is supposed to be a continuation of the former,
- finally, if  $\text{sta-1n} = \text{sta-21}$ , then the second computation continues the first computation.

As is easy to check, our composition is associative, and the following equations are satisfied:

|                         |        |  |
|-------------------------|--------|--|
| $() \bullet \text{com}$ | $= ()$ | for any $\text{com} : \text{Co.State}$   |
| $\text{com} \bullet ()$ | $= ()$ | for any $\text{com} : \text{FiCo.State}$ |



$$\text{com} \bullet () = \text{com} \quad \text{for any } \text{com} : \text{InCo.State}$$

By a *bundle of computations* we mean any set of computations which includes the empty computation. We shall denote:

$$\begin{aligned} \text{Bun.State} &= \{ P \mid P \subseteq \text{Co.State} \text{ and } () : P \}^{104} \\ -A &= \text{State} - A && \text{— for any set of states } A \subseteq \text{State} \\ [A] &= \{ (\text{sta} \mid \text{sta} : A) \mid \{ () \} \} && \text{— for any set of states } A \subseteq \text{State}; [A] \text{ is called a } \textit{test} \\ \{P\} &&& \text{— the set of all states that appear in bundle } P, \text{ e.g. } \{[A]\} = A \\ \text{lb} &= [\text{State}] && \text{— } \textit{identity bundle} \\ \text{Eb} &= \{ () \} && \text{— } \textit{empty bundle} \\ \text{fin.P} &= P \cap \text{State}^{c*} && \text{— } \textit{finitistic part of } P \\ \text{inf.P} &= (P - \text{fin.P}) \mid \{ () \} && \text{— } \textit{infinitistic part of } P \end{aligned}$$

In our new model bundles will represent denotations of programs. A bundle is said to be *infinitistic* if it includes infinite computations. The *composition of bundles* is defined as follows:

$$P \bullet Q = \{ \text{com-1} \bullet \text{com-2} \mid \text{com-1} : P \text{ and } \text{com-2} : Q \}$$

As is easy to check:

$$(\text{Bun.State}, \subseteq, \bullet, \text{Eb}, \text{lb})$$

is a set-theoretic quasinet<sup>105</sup>, but not a net since

$$P \bullet \text{Eb} = \text{inf.P}$$

Similarly as in the case of languages, quasinet of bundles constitute complete lattices.

In obvious contexts we shall allow to omit  $\bullet$  and write  $\text{com-1com-2}$  and  $PQ$  respectively. We assume also that composition binds stronger than union, i.e.

$$PQ \mid RF = (PQ) \mid (RF)$$

Empty bundle  $\text{Eb}$  represents a program that “can’t run”. A union of bundles  $P \mid Q$  represents a (possibly) non-deterministic branching of programs  $P$  and  $Q$ , and a composition  $PQ$  represents a sequential composition of these programs. A computation consisting of two states only ( $\text{sta-1}$ ,  $\text{sta-2}$ ) is called an *atomic computation*. A bundle whose all non-empty computations are atomic is called an *atomic bundle*.

The powers of a bundles —  $P^n$ ,  $P^+$  and  $P^*$  — are defined according to the rule indicated for abstract quasinet (Sec. 12.2.1). As is easy to see:

$$P^+ = \bigcup \{ P^n \mid n = 1, 2, \dots \}.$$

It should be observed that in the quasinet of bundles we have so called *zero divisors*, i.e., such  $P$  and  $Q$  both different from  $\text{Eb}$ , that

$$PQ = \text{Eb}.$$

E.g., if  $A \cap B = \{ \}$ , then  $[A][B] = \text{Eb}$ . As is easy to check (see [21] and [24]) the following equations are true for any bundles  $P$ ,  $Q$  and  $R$ :

$$\begin{aligned} [A]P &= \{ \text{com} \mid \text{com} : P \text{ and } \text{first.com} : A \} \mid \{ () \} \\ P(Q \mid R) &= PQ \mid PR && \text{— } \textit{left distributivity over union} \\ (Q \mid R)P &= QP \mid RP && \text{— } \textit{right distributivity over union} \end{aligned}$$

In the quasinet of bundles we can define constructors that correspond to structured constructors of programs:

$$P ; Q = PQ$$

<sup>104</sup> Bundles will be denoted by  $P$ ,  $Q$ ,  $R$ ,... and sets of states by  $A$ ,  $B$ ,  $C$ ,...

<sup>105</sup> Note that if we had not assumed that all bundles include empty computation, then the composition of an infinitistic bundle with the empty bundle would be the empty bundle, which would mean that if we compose sequentially a program which may loop indefinitely with a program that can’t run, then the composed program can’t run.

$$\text{if } (T, F) \text{ then } P \text{ else } Q \text{ fi} = TP \mid FQ$$

In the second equation we assume that  $T, F \subseteq [\text{State}]$  are disjoint subsets of identity bundle that represent a three-valued predicate. Note that if we consider a bundle of the form

$$R ; \text{if } (T, F) \text{ then } P \text{ else } Q \text{ fi} = R (TP \mid FQ)$$

then a computation of  $R$  which terminates with a state in  $T$  composed with a computation of  $FQ$  results an empty computations, because it “cannot happen”.

To deal with infinite computations we introduce a composition of an infinite sequence of bundles, informally:

$$P_1 \bullet P_2 \bullet \dots$$

To define this operation we need a few auxiliary concepts. We say that **com-1** is a *prefix* of **com-2**, in symbols

$$\text{com-1} \sqsubseteq \text{com-2}$$

if there exists **com**, such that  $\text{com-2} = \text{com-1} \bullet \text{com}$ . Note that

$$() \sqsubseteq ()$$

but

$$() \sqsubseteq \text{com} \quad \text{does not hold for } \text{com} \neq (). \quad (12.2.2-1)$$

A composition of an infinite sequence of computations denoted by

$$C.(\text{com-1}, \text{com-2}, \dots)$$

is the shortest<sup>106</sup> **com** such that  $(\forall n) (\text{com-1} \bullet \dots \bullet \text{com-n} \sqsubseteq \text{com})$ . Note that if for some  $n \geq 1$ ,

$$\text{com-1} \bullet \dots \bullet \text{com-n} = (),$$

then

$$C.(\text{com-1}, \text{com-2}, \dots) = ().$$

As we see, a composition of an infinite sequence of computations may be empty, finite or infinite. Now, we can define a composition of an infinite sequence of bundles:

$$C.(P_1, P_2, \dots) = \{ C.(\text{com-1}, \text{com-2}, \dots) \mid (\forall i) \text{com-i} : P_i \}$$

and an infinite power of a bundle

$$P^\infty = C.(P, P, \dots)$$

Using the introduced notation we can define the denotation of a while-loop:

$$\text{while } (T, F) \text{ do } P \text{ od} = (TP)^*F \mid (TP)^\infty$$

In this case  $(TP)^*F$  is the least solution of the equation

$$X = TPX \mid F$$

which can be easily proved using Kleene’s theorem 12.2.1-1. At the same time, if all nonempty computations in  $(TP)^\infty$  are infinite, then  $(TP)^\infty$  is the greatest solution of the equation

$$X = TPX$$

Proof in [24].

<sup>106</sup> Note that if starting with some  $n$  all  $\text{com}_i$  are tests then  $C.(\text{com}_1, \text{com}_2, \dots)$  is finite and is either empty or equal to  $\text{com}_1 \bullet \dots \bullet \text{com}_{n-1}$ .

### 12.2.3 Strong total correctness of bundles

Bundles constitute an adequate framework for the development of a Hoare-like correctness theory for strong total correctness with clean termination, i.e. without abortion (cf. Sec. 8.7). In **Lingua**, abortion states carry an error message in their error registers and in this way indicate that a program has stopped its execution and displays an error message. In the current abstract model of states we shall only assume the existence of a distinguished subset of the set of states whose elements are called *abortion states*.

$\text{Abort} \subseteq \text{State}$

We will say that a program terminates its execution *cleanly* if its terminal state is not an abortion state.

Now, let  $A$  be an arbitrary subset of  $\text{State}$ , and let  $P$  be an arbitrary bundle over  $\text{State}$ . We define two operations of *left-* and respectively *right composition* of a bundle with a set of states.

$$\begin{aligned} A \bullet P &= \{ \text{sta-}n \mid (\exists (\text{sta-}1, \dots, \text{sta-}n) : P) \text{ sta-}1 : A \} \\ P \bullet B &= \{ \text{sta-}1 \mid (\exists (\text{sta-}1, \dots, \text{sta-}n) : P) \text{ sta-}n : B \} \end{aligned}$$

Here we use the same symbol as for the composition of bundles and we allow omitting this symbol thus writing  $AP$  and  $PB$ . As is easy to prove both these operations are monotone and associative.

**Lemma (12.2.3-1)** *For any  $A, B \subseteq \text{State}$  and any  $P, Q : \text{Bun.State}$*

$$\begin{aligned} &\text{if } A \subseteq B \text{ then } AP \subseteq BP \text{ and } PA \subseteq PB \\ &\text{if } P \subseteq Q \text{ then } AP \subseteq AQ \text{ and } PA \subseteq QA \\ &A(PQ) = (AP)Q \text{ and } (PQ)A = P(QA) \end{aligned}$$

■

Using our operations we can define the properties of partial correctness and of weak total correctness of bundles:

$AP \subseteq B$  — *partial correctness* of  $P$  for precondition  $A$  and postconditions  $B$ ; every finite computation of  $P$  that starts in  $A$ , terminates in  $B$ , but there may be infinite computations that start in  $A$ .

This property corresponds to a partial correctness in the sense of C.A.R. Hoare (cf. [5] and [61]).

$A \subseteq PB$  — *weak total correctness* of  $P$  for precondition  $A$  and postconditions  $B$ , for every  $a : A$  there exists a computation in  $P$  that starts with  $a$ , and terminates in  $B$ .

This property is called weak total correctness, since it only guarantees that for any  $s$  in  $A$  there exists a finite computation that starts with  $s$  and terminates in  $B$ , but there may be other computations that start with  $s$ , but either terminate outside  $B$  or do not terminate at all<sup>107</sup>. Of course, in the case of deterministic programs weak total correctness is just total correctness, and if we assume that  $B$  does not include states that carry an error message, then it is a *clean total correctness*. In the deterministic case clean total correctness implies partial correctness.

In the case of nondeterministic programs with possibly infinite computations we need a stronger concept of correctness which would guarantee that all computations that start in  $A$  terminate in  $B$ . To define this concept we introduce a *strong composition* of a bundle with a set of states:

$$P \blacksquare B = \{ \text{sta} \mid (\forall \text{com} : P) \text{ if first.com} = \text{sta} \text{ then } \text{com} : \text{State}^{c*} \text{ and last.com} : B \}$$

where  $\text{first.com}$  and  $\text{last.com}$  are the first and the last element of  $\text{com}$ .

$P \blacksquare B$  is the set of all states which give rise to finite computations only, and all these computations terminate in  $B$ . Consequently, if  $\text{sta} : P \blacksquare B$  then all computations that start with  $\text{sta}$  are finite and terminate in  $B$ . Similarly to the properties (12.2.3-1) also now we have monotonicity and associativity:

**Lemma (12.2.3-2)** *For any  $A, B \subseteq \text{State}$  and any  $P, Q : \text{Bun.State}$*

$$\begin{aligned} &\text{if } A \subseteq B \text{ then } P \blacksquare A \subseteq P \blacksquare B \\ &\text{if } P \subseteq Q \text{ then } P \blacksquare A \subseteq Q \blacksquare A \\ &(PQ) \blacksquare A = P \blacksquare (Q \blacksquare A) \end{aligned}$$

■

<sup>107</sup> The idea to call this total correctness a weak total correctness is due to Krzysztof Apt (personal communication).

Now, the strong correctness is defined as follows:

$A \subseteq P \blacksquare B$  — *strong correctness* of  $P$  for precondition  $A$  and postcondition  $B$ ; all computations of  $P$  that start in  $A$  terminate in  $B$ .

If  $B$  includes no error states then we say that  $P$  is strongly correct with *clean termination*. Of course, in a deterministic case weak total correctness is equivalent to strong correctness, but in a nondeterministic case the former is weaker than the latter. The following obvious lemma is useful in developing proof rules for structured constructors:

**Lemma 12.2.3-3** For any  $A, B \subseteq \text{State}$  and  $P : \text{Bun.State}$

$A \subseteq P \blacksquare B$  iff  $AP \subseteq B$  and  $A \subseteq P \blacksquare \text{State}$  ■

Having defined strong total correctness of bundles we can formulate Hoare-like proof rules similar as in the case of binary relations (cf. Sec. 8.7).

**Lemma 12.2.3-4** For any  $A, D \subseteq \text{State}$  and  $P, Q : \text{Bun.State}$

$$\begin{array}{l} \uparrow \text{there exist } B, C \subseteq \text{State such that} \\ (1) A \subseteq P \blacksquare B \\ (2) B \subseteq C \\ (3) C \subseteq Q \blacksquare D \\ \hline (4) A \subseteq (PQ) \blacksquare D \end{array}$$

**Proof** If (1) – (3) are satisfied then partial correctness  $A(PQ) \subseteq D$  is immediate from Lemmas 12.2.3-1 and 12.2.3-3. The termination is obvious. In turn, if (4) is satisfied then by Lemma 12.2.3-2,  $A \subseteq P \blacksquare (Q \blacksquare D)$  and setting  $B = C = Q \blacksquare D$  we get the proof.

■

**Lemma 12.2.3-5** For any predicate  $(T, F)$  any  $A, B \subseteq \text{State}$  and  $P, Q : \text{Bun.State}$

$$\begin{array}{l} \uparrow \begin{array}{ll} (1) A \cap \{T\} & \subseteq P \blacksquare B \\ (2) A \cap \{F\} & \subseteq Q \blacksquare B \\ (3) A & \subseteq \{T\} \mid \{F\} \end{array} \\ \hline (4) A & \subseteq \text{if } (T, F) \text{ then } P \text{ else } Q \text{ fi} \blacksquare B \end{array}$$

Since

$\text{if } (T, F) \text{ then } P \text{ else } Q \text{ fi} = TP \mid FQ,$

the proof is analogous to the former. Of course, in the case of classical predicates  $T \mid F = \text{State}$ , and therefore condition (3) is a tautology and may be omitted.

We skip the discussion of a while-loop to avoid technicalities that would go beyond the scope of this section. Such rules for the case of I-O functions have been discussed in [28] and in Sec. 8.7.2.

Two important issues have to be pointed out at the end. First concerns the fact that we are not building here any formalized logic of programs like in [61] or [4]. Our proof rules are just lemmas proved on the ground of set theory to be used in proving properties of programs on the same ground. Logic involved in these proofs is just a usual mathematical logic.

The second issue concerns the fact that our proof rules may be regarded as construction rules of correct programs analogously as in Sec. 9.4.

## 12.2.4 Temporal quantifiers

*Temporal quantifiers* may be easily defined in the model of bundles. For that sake let us regard computations as functions that map nonnegative integers into states. We assume further that each such function is defined on an initial interval of the form  $[1, \dots, n]$ . By  $\text{dom.com}$ , where  $\text{com}$  is a computation, we shall denote the domain of  $\text{com}$ . Let  $A, B \subseteq \text{State}$  represent the truth parts of a predicate, let  $\text{com}$  be an arbitrary computation, and let  $i$  and  $j$  run over  $\text{dom.com}$ . Then:

|                                |     |  |                              |
|--------------------------------|-----|--|------------------------------|
| $A \sqcap \text{com}$          | iff | $(\forall i) \text{com}.i : A$   | — <i>always A</i>            |
| $A \diamond \text{com}$        | iff | $(\exists i) \text{com}.i : A$   | — <i>eventually A</i>        |
| $(A \cup B) \text{com}$        | iff | $(\exists j) ((\forall i \leq j) \text{com}.i : A) \text{ and } \text{com}.j : B$                            | — <i>A until B</i>           |
| $(A \mathcal{W} B) \text{com}$ | iff | $(A \cup B) \text{com} \text{ or } (\text{not } (B \diamond \text{com}) \text{ and } (A \sqcap \text{com}))$ | — <i>A unless B</i>          |
| $(A \rightarrow B) \text{com}$ | iff | <b>if</b> $(\exists i) \text{com}.i : A$ <b>then</b> $(\exists j \geq i) \text{com}.j : B$                   | — <i>if A then later B</i>   |
| $(A \leftarrow B) \text{com}$  | iff | <b>if</b> $(\exists i) \text{com}.i : B$ <b>then</b> $(\exists j \leq i) \text{com}.j : A$                   | — <i>B only if earlier A</i> |

The quantifier  $\mathcal{W}$  is called by Mordechai Ben-Ari in [13] a *weak until* because it does not require that  $B$  eventually becomes true. In such a case  $A$  remains true forever. Our quantifiers may be easily generalized to bundles:

$$A \sqcap P \text{ iff } (\forall \text{com} : P) A \sqcap \text{com}$$

and analogously for other quantifiers. We say that  $A$  is *hereditary* in a bundle  $P$ , in symbols

$$A \blacktriangleright P$$

if

$$(\forall \text{com} : P) \text{ either } \neg A \sqcap \text{com} \text{ or } (\exists i) (\forall j \geq i) (\text{com}.j : A)$$

## 12.3 Petri nets and trace languages

### 12.3.1 Trace languages of Antoni Mazurkiewicz

Let  $\text{Alp}$  be a finite or infinite alphabet. By a *dependency relation* or simply a *dependency* in  $\text{Alp}$  we mean any finite binary relation  $D \subseteq \text{Alp} \times \text{Alp}$  such that, if  $(a, b) : D$  then  $(b, a)$  and  $(a, a) : D$ . With every dependency we associate its alphabet  $\text{alp}.D$  which is the set of all letters that appear in  $D$ . Consequently, a dependency  $D$  is reflexive and symmetric in  $\text{alp}.D$ . A dependency  $D$  is said to be a *full dependency* if

$$D = (\text{alp}.D)^{c2}$$

By  $\text{Dep.Alp}$  we shall denote the set of all dependencies over  $\text{Alp}$ . Dependencies have the following important properties:

1. empty relations, identity relations and full relations in their alphabets are dependencies,
2. union and intersection of a finite number of dependencies is a dependency,
3. every dependency is a finite union of full dependencies.

For instance,

$$D = \{a, b\}^{c2} \mid \{a, c\}^{c2}$$

is a dependency relation over  $\text{alp}.D = \{a, b, c\}$ . By an *independency relation induced by  $D$*  we mean a relation

$$\text{ind}.D = (\text{alp}.D)^{c2} - D.$$

Clearly every independency is symmetric and irreflexive in its alphabet  $\text{alp}.(\text{ind}.D)$ . In the case of our example  $\text{ind}.D = \{(b, c), (c, b)\}$ , and  $\text{alp}.(\text{ind}.D) = \{b, c\}$ . Independency relations may be described as a symmetric closures, abbreviated *sc*, of a non-symmetric relations. E.g.,

$$\text{ind}.D = \text{sc}.\{(b, c)\}.$$

The alphabet  $\text{alp.}(\text{ind.}D)$  may be a proper subset of  $\text{alp.}D$ , but may also be equal to  $\text{alp.}D$ . E.g. if  $D$  is an identity relation, and  $\text{alp.}D$  includes at least two elements, then  $\text{alp.}D = \text{alp.}(\text{ind.}D)$ .

Given a dependency  $D$  we define an equivalence relation  $\equiv_D$  between words over  $\text{alp.}D$  as the least congruence in the monoid of these words such that for any  $a, b : \text{alp.}D$ :

$$\text{if } (a, b) : \text{ind.}D \text{ then } ab \equiv_D ba \quad (12.3.1-1)$$

This relation is called *trace equivalence* for  $D$ . Equivalence classes over  $\equiv_D$  in  $(\text{alp.}D)^*$  are called *traces* over  $D$ , and constitute a quotient monoid  $(\text{alp.}D)^*/\equiv_D$ . An element of that monoid represented by a word  $w$  will be denoted by  $[w]_D$  or simply by  $[w]$  if  $D$  is understood. In that case  $w$  is said to be a *representant* of  $[w]$ . In the general case a trace may have many representants, but

**Fact 12.3.1-1** If a dependency  $D$  is full, then each  $D$ -trace has exactly one representant.

By

$$\text{Tra.}D = \{[w] \mid w : (\text{alp.}D)^*\}$$

we denote the set of all traces over  $D$ . If by  $\bullet$  we denote both the concatenation of words and the concatenation of traces then the latter is defined as follows:

$$[w_1]_D \bullet [w_2]_D = [w_1 \bullet w_2]_D$$

where  $w_1, w_2$  are words over  $\text{alp.}D$ . By the definition of traces,  $[w]$  is the set of all words that arise from  $w$  by the permutations of all adjacent independent letters. For instance, in the case of our example

$$[\text{abbca}] = \{\text{abbca}, \text{abcba}, \text{acbba}\}.$$

For any dependency  $D$ , by a *trace language* over  $D$  we mean any set of traces over  $D$ . From now on, “usual” languages described in Sec. 2.5 will be called *word languages* or just *languages*. For any word language  $L$  over  $\text{alp.}D$  we define the corresponding trace language

$$[L]_D = \{[w]_D \mid w : L\}$$

In this case  $L$  is called a *representant* of  $[L]_D$ . Of course, if  $D$  is not a full relation then  $[L]_D$  has more than one representant.

If  $D$  is understood, then we simply write  $[L]$ . By  $\text{TraLan.}Alp$  we shall denote the set of all trace languages over  $Alp$ , i.e.,

$$\text{TraLan.}Alp = \{[L]_D \mid L : \text{Lan.}Alp, D : \text{Dep.}Alp\}$$

Now, with every trace language  $TL$  we can assign a word language which includes all words belonging to the traces of  $TL$ . A function which transforms trace languages into word languages is the following

$$\begin{aligned} T2L : \text{TraLan.}Alp &\mapsto \text{Lan.}Alp \\ T2L.TL &= \bigcup \{t \mid t : TL\} \end{aligned}$$

Since traces are classes of abstraction, hence sets, their unions makes sense. Note that

$$L \subseteq T2L.[L]_D \quad (12.3.1-2)$$

but  $T2L.[L]_D$  may be significantly larger than  $L$  because it includes all words created by the permutations of adjacent independent letters in the words of  $L$ . Language  $T2L.[L]_D$  is called the *D-completion* of  $L$ .

**Fact 12.3.1-2** For every  $L$  and  $D$ ,  $T2L.[L]_D$  is the largest representant of  $[L]_D$ .

**Fact 12.3.1-3** For every  $L$  and  $D$ ,  $T2L.[T2L.[L]_D]_D = T2L.[L]_D$

If

$$L = T2L.[L]_D$$

then  $L$  is said to be *D-complete*. Consider the following example :

$$L = \{\text{abbca}, \text{abc}\}$$

$$D = \{a, b\}^{c^2} \mid \{a, c\}^{c^2} \text{ hence } \text{Ind} = \{(b, c), (c, b)\}$$

In this case

$$\begin{aligned} [L] &= \{ [abbca], [abc] \} \\ [abbca] &= \{ abbca, abcba, acbba \} \\ [abc] &= \{ abc, acb \} \\ T2L.[L] &= \{ abbca, abcba, acbba, abc, acb \} \end{aligned}$$

**Fact 12.3.1-4** If  $D$  is full than any word language over  $\text{alp}.D$  is  $D$ -complete

The last fact implies that trace languages are natural generalizations of word languages. If  $D$  is full then  $[L]_D$  is said to be  $D$ -sequential. Of course, each  $D$ -sequential language is  $D$ -complete but not vice versa.

For every dependency relation  $D$ , the set of all trace languages over  $D$ , i.e.,

$$\text{TraLan}.D = \{ [L]_D \mid L : \text{Lan}(\text{alp}.D) \}$$

constitutes a monoid with the operation

$$[L_1] \bullet [L_2] = \{ [w_1] \bullet [w_2] \mid w_1 : L_1, w_2 : L_2 \}$$

and unit  $[()] = \{()\}$ . We shall allow writing  $[L_1][L_2]$  for  $[L_1] \bullet [L_2]$  and analogously for traces. We also set:

$$\begin{aligned} [L]^0 &= [()] \\ [L]^{n+1} &= L^n \bullet [L] \text{ for } n = 0, 1, \dots \\ [L]^* &= \bigcup \{ [L]^n \mid n = 0, 1, \dots \} \end{aligned}$$

**Theorem 12.3.1-1** (see [76]) For any dependency relation  $D$ , any word languages  $L_1$  and  $L_2$  over  $\text{alp}.D$ , and any family of word languages  $\{L_i \mid i = 1, 2, \dots\}$  over  $\text{alp}.D$  the following properties hold:

- (1)  $\{ \}$  =  $\{ \}$
- (2)  $[L_1][L_2]$  =  $[L_1 L_2]$
- (3)  $[L_1] \mid [L_2]$  =  $[L_1 \mid L_2]$
- (4)  $\bigcup \{ [L_i] \mid i = 1, 2, \dots \}$  =  $[\bigcup \{ L_i \mid i = 1, 2, \dots \}]$
- (5)  $[L]^*$  =  $[L^*]$
- (6) if  $L_1 \subseteq L_2$  then  $[L_1] \subseteq [L_2]$  ■

Note that in (6) a converse implication does not need to be true. E.g.

$$\{ [abbca, abcba] \} \subseteq \{ [abbca] \}$$

It holds, however, if  $L_1$  and  $L_2$  are the largest representants of respective trace languages. Indeed, let  $w : L_1$ . In that case  $[w] : [L_2]$ . Let  $[w] = \{w_1, \dots, w_n\}$ . Since  $L_2$  is the largest representant of  $[L_2]$ , all  $w_i$ 's must belong to  $L_2$ , and, therefore,  $w : L_2$ .

**Theorem 12.3.1-2** For any dependency relation  $D$  the set  $\text{TraLan}.D$  of all trace languages over this relation is a set-theoretic net with concatenation of trace languages as the monoid operation, with the unit of the monoid  $[()]$  and the least element  $\{ \}$ . ■

**Proof** First note the following facts:

- the fact that  $(\text{TraLan}.D, \bullet, [()])$  is a monoid is obvious from (2) of Theorem 12.3.1-1,
- the fact that  $(\text{TraLan}.D, \subseteq)$  is a CPO, follows from (4),
- the equalities  $[L][()] = [()][L]$  follow from (1) and (2).

To prove that  $\bullet$  is continuous — i.e. that it is continuous in each of its arguments separately — consider a chain of trace languages:

$$[L_1] \subseteq [L_2] \subseteq \dots$$

where all  $L_i$ 's are the largest representants of the respective trace languages. Let  $Q$  be the largest representant of  $[Q]$ . To prove that  $\bullet$  is continuous in second argument we have to prove that

$$[Q] \bullet \bigcup \{ [L_i] \mid i = 1, 2, \dots \} = \bigcup \{ [Q] \bullet [L_i] \mid i = 1, 2, \dots \}$$



Indeed, by (2) and (4) and the continuity of the concatenations of word languages, we have

$$\begin{aligned} [Q] \bullet U \{[L_i] \mid i = 1, 2, \dots\} &= [Q] \bullet [U \{L_i \mid i = 1, 2, \dots\}] = [Q \bullet U \{L_i \mid i = 1, 2, \dots\}] = \\ [U \{Q \bullet L_i \mid i = 1, 2, \dots\}] &= U \{[Q \bullet L_i] \mid i = 1, 2, \dots\} = U \{[Q] \bullet [L_i] \mid i = 1, 2, \dots\}. \end{aligned}$$

The proof for the first argument is analogous. ■

Now, we can proceed to the definitions of synchronization operations of word languages and trace languages. We start from the notion of projection. Let  $B \subseteq \text{Alp}$ , and let  $w$  be a word over  $\text{Alp}$ . By a *projection* of a word  $w$  over the alphabet  $B$  we mean a word over  $B$  defined as follows:

```

pro.(B, Alp) : Word.Alp  $\mapsto$  Word.B
pro.(B, Alp).w =
  w = ()  $\rightarrow$  ()
  let
    z  $\bullet$  a = w    where a : Alp
    a /: B  $\rightarrow$  pro.(B, Alp).z
    a : B  $\rightarrow$  (pro.(B, Alp))  $\bullet$  a

```

Projection function removes from  $w$  all letters which are not in  $B$ . This function may be extended to languages in an obvious way:

$$\text{pro.}(B, \text{Alp}).L = \{\text{pro.}(B, \text{Alp}).w \mid w : L\}$$

Let now  $C$  and  $D$  be dependencies and let  $C \subseteq D$ . By a *trace projection* of a trace  $t$  on dependency  $C$  we mean a trace over  $C$  defined as follows:

```

tr-pro.(C, D) : Tra.D  $\mapsto$  Tra.C
tr-pro.(C, D).t =
  t = [()]_D  $\rightarrow$  [()]_C
  let
    p  $\bullet$  [a]_D = t    where a : alp.D
    a /: alp.C  $\rightarrow$  tr-pro.(C, D).p
    a : alp.C  $\rightarrow$  (tr-pro.(C, D).p)  $\bullet$  [a]_C

```

This function satisfies the following equality for any  $w : \text{alp.D}$ :

$$\text{tr-pro.}(C, D).[w]_D = [\text{pro.}(\text{alp.C}, \text{alp.D}).w]_C$$

As we see, trace projection given a trace over  $D$  removes from its representants all letter which are not in  $\text{alp.C}$  and restricts the dependency to  $C$ . Here two cases are possible.

If  $\text{alp.C}$  is strictly included in  $\text{alp.D}$ , then traces over  $C$ , regarded as sets, are not comparable with traces over  $D$ .

### Example 12.3.1-1

Let

$$\begin{aligned} D &= \{a, b\}^{c^2} \mid \{a, c\}^{c^2} \mid \{a, d\}^{c^2} \text{ and then } \text{ind.D} = \{ (b, c), (c, b), (b, d), (d, b), (c, d), (d, c) \} \\ C &= \{a, b\}^{c^2} \mid \{a, c\}^{c^2} \text{ and then } \text{ind.C} = \{ (b, c), (c, b) \} \end{aligned}$$

In this case  $\text{ind.C} \subseteq \text{ind.D}$  and, e.g.

$$\begin{aligned} [abcd]_D &= \{abcd, abdc, acbd, acdb, adcb, adbc\}. \\ \text{tr-pro.}(C, D).[abcd]_D &= [abc]_C = \{abc, acb\} \end{aligned}$$

If, however,  $\text{alp.C} = \text{alp.D}$  then  $\text{ind.C}$  may be larger than  $\text{ind.D}$  since  $C$  gives “more freedom” for the permutation of letters. E.g.:

$$\begin{aligned} D &= \{a, b\}^{c^2} \mid \{a, c\}^{c^2} \mid \{c, b\}^{c^2} = \{a, b, c\}^{c^2} & \text{and then } \text{ind.D} &= \{ \} \\ C &= \{a, b\}^{c^2} \mid \{a, c\}^{c^2} & \text{and then } \text{ind.C} &= \{(b, c), (c, b)\}. \end{aligned}$$

In this case

$$[abc]_D = \{abc\}$$

$$\text{tr-pro.}(C, D).[abc]_D = [abc]_C = \{abc, acb\}$$

and

$$[abc]_D \subseteq \text{tr-pro.}(C, D).[abc]_D$$

### Example 12.3.1-2

Let

$$D = \{a, b\}^{c^2} \mid \{a, c\}^{c^2}, \quad \text{alp.D} = \{a, b, c\}, \quad \text{ind.D} = \{(b, c)\}$$

$$C = \{b\}^{c^2} \mid \{c\}^{c^2}, \quad \text{alp.C} = \{a, c\}, \quad \text{ind.C} = \{(b, c)\}$$

In this case

$$[bac]_D = \{bac\}$$

$$\text{tr-pro.}(C, D).[bac]_D = [bc]_D = \{bc, cb\}$$

Poprosilem Andrzeja o uzupełnienie. ???

Consider now two word languages  $L_1$  and  $L_2$ . By a *synchronization* of these languages we mean a word language over  $\text{Alp} = \text{alp.L}_1 \mid \text{alp.L}_2$  defined as follows:

$$L_1 \parallel L_2 = \{w \mid w : \text{Alp}^*, \text{pro.}(\text{alp.L}_i, \text{Alp}).w : L_i, i = 1, 2\}$$

**Theorem 12.3.1-3** (see [76]) The synchronization of word languages is commutative, associative and distributive over arbitrary unions, i.e. for any word languages  $L, L_1, L_2, L_3$ , and any family of word languages  $\{L_i \mid i : \text{Ind}\}$  over a common alphabet:

$$L_1 \parallel L_2 = L_2 \parallel L_1$$

$$L_1 \parallel (L_2 \parallel L_3) = (L_1 \parallel L_2) \parallel L_3$$

$$(U \{L_i \mid i : \text{Ind}\}) \parallel L = U \{L_i \parallel L \mid i : \text{Ind}\}$$

■

Consider two trace languages  $\text{TL}_1$  and  $\text{TL}_2$  over two dependencies  $D_1$  and  $D_2$  respectively. By a *synchronization* of these trace languages we mean a trace language over alphabet  $\text{Alp} = \text{alp.D}_1 \mid \text{alp.D}_2$  and dependency  $D = D_1 \mid D_2$  defined as follows

$$\text{TL}_1 \parallel \text{TL}_2 = \{[w] \mid w : \text{Alp}^*, \text{tr-pro.}(D_i, D).t : \text{TL}_i, i = 1, 2\}$$

**Theorem 12.3.1-4** (see [76]) For any dependencies  $D_1$  and  $D_2$ , and any word languages  $L_1$  and  $L_2$  over  $\text{alp.D}_1$  and  $\text{alp.D}_2$  respectively the following equality holds:

$$[L_1]_{D_1} \parallel [L_2]_{D_2} = [L_1 \parallel L_2]_{D_1 \mid D_2}$$

■

**Theorem 12.3.1-5** (see [76]) The synchronization of trace languages is commutative, associative and distributive over arbitrary unions, i.e. for any trace languages  $\text{TL}, \text{TL}_1, \text{TL}_2, \text{TL}_3$ , and any family of trace languages  $\{\text{TL}_i \mid i : \text{Ind}\}$  over a common alphabet:

$$\text{TL}_1 \parallel \text{TL}_2 = \text{TL}_2 \parallel \text{TL}_1$$

$$\text{TL}_1 \parallel (\text{TL}_2 \parallel \text{TL}_3) = (\text{TL}_1 \parallel \text{TL}_2) \parallel \text{TL}_3$$

$$(U \{\text{TL}_i \mid i : \text{Ind}\}) \parallel \text{TL} = U \{\text{TL}_i \parallel \text{TL} \mid i : \text{Ind}\}$$

■

## 12.3.2 Trace languages and Petri Nets

One of the most important fields of applications of trace languages are Petri nets. Trace languages play the same role for Petri nets, as regular word languages for iterative programs (programs without recursive procedures). Below we sketch this construction following again [76]. By a *Petri net* we mean a quadruple

$\text{net} = (\text{Place}, \text{Transition}, \text{Flow}, \text{Ini})$

where

$\text{pla} : \text{Place}$

a finite set of elements called *places*

$\text{tra} : \text{Transition}$

a finite set of elements called *transitions*

$\text{Flow} \subseteq \text{Place} \times \text{Transition} \mid \text{Transition} \times \text{Place}$

*flow relation*

$\text{Ini} \subseteq \text{Place}$

*initial marking*

and where the following assumptions are satisfied

$\text{Place} \cap \text{Transition} = \{\}$

$\text{Flow} \cap \text{Flow}^{-1} = \{\}$

no place is an entry and an exit of the same transition

$\text{dom.Flow} \mid \text{cod.Flow} = \text{Pla} \mid \text{Tra}$

there are no isolated places or transitions

Now, let for any  $\text{tra} : \text{Transition}$

$\text{Entry.tra} = \{\text{pla} \mid (\text{pla}, \text{tra}) : \text{Flow}\}$

*entries* of transition  $\text{tra}$

$\text{Exit.tra} = \{\text{pla} \mid (\text{tra}, \text{pla}) : \text{Flow}\}$

*exits* of transition  $\text{tra}$

$\text{Neigh.tra} = \text{Entry.tra} \mid \text{Exit.tra}$

*neighborhood* of transition  $\text{tra}$

By a *marking* of a net we mean any set of places of this net including an empty set:

$\text{mar} : \text{Marking} = \text{Sub.Place}$

With every transition we assign a partial *transition function* of the net which describes the transformation of an input marking into an output marking that occurs when this transition is executed (fired).

$\text{Tf} : \text{Transition} \mapsto \text{Marking} \rightarrow \text{Marking}$

A. Mazurkiewicz defines this function in the following way:

$\text{Tf.tra.mar}_{\text{in}} = \text{mar}_{\text{fi}}$  iff

( $\text{in}$  – initial,  $\text{fi}$  – final)

$\text{Entry.tra} \subseteq \text{mar}_{\text{in}}$  **and**

all entries of  $\text{tra}$  are marked

$\text{Exit.tra} \cap \text{mar}_{\text{in}} = \{\}$  **and**

no exit of  $\text{tra}$  is marked

$\text{mar}_{\text{fi}} = \text{Exit.tra} \mid (\text{mar}_{\text{in}} - \text{Entry.tra})$

An explicit definition of this function may be the following

$\text{Tf.tra.mar} =$

$\text{Entry.tra} \subseteq \text{mar}$  **and**  $\text{Exit.tra} \cap \text{mar} = \{\}$   $\rightarrow \text{Exit.tra} \mid (\text{mar} - \text{Entry.tra})$

**true**

$\rightarrow ?$

As we see, if all entry places of  $\text{tra}$  are marked, and no exit places of  $\text{tra}$  are marked, then the output marking exists and consists of all exit places of  $\text{tra}$  plus the “unused” places of the initial marking, i.e.,  $\text{mar} - \text{Entry.tra}$ . In that case we say that marking  $\text{mar}$  *enables transition tra*.

Two particular cases of this definition are to be pointed out.

If  $\text{Entry.tra} = \{\}$  — in this case we say that  $\text{tra}$  is an *orphan* — then  $\text{tra}$  is enabled by any marking  $\text{mar}$  disjoint with  $\text{Exit.tra}$  including the empty marking. In this case

$\text{Tf.tra.mar} = \text{Exit.tra} \mid \text{mar}$

$\text{Tf.tra.}\{\} = \text{Exit.tra}$

The interpretation of these facts are quite natural. Entry and exit places of a transit define a necessary and sufficient condition to fire that transit. This condition allows for a firing of a transition if all entry places are marked and no exit places are marked. No entry places and empty marking (hence no marks in exit places) means that the firing condition is satisfied.

The second case to be considered is when  $\text{Exit.tra} = \{\}$ . In that case  $\text{tra}$  is said to be a *widow*, and for any marking  $\text{mar}$  which satisfies  $\text{Entry.tra} \subseteq \text{mar}$  we have

$\text{Tf.tra.mar} = \text{mar} - \text{Entry.tra}$

In particular

$$\text{Tf.tra.}(\text{Entry.tra}) = \{\}$$

By a *path of transitions* or simply a *path*, we mean any finite, possibly empty, sequence of transitions:

$$\text{pat} : \text{Path} = \text{Transition}^c^*$$

Now, we can generalize the single-step function  $\text{Tf}$  to a many-steps function called the *reachability function* and defined as follows:

$$\text{Rf} : \text{Path} \mapsto \text{Marking} \rightarrow \text{Marking}$$

$$\begin{aligned} \text{Rf.}(\text{tra}_1, \dots, \text{tra}_n).\text{mar} = \\ n = 0 & \rightarrow \text{mar} \\ \text{true} & \rightarrow \text{Tf.tra}_n.(\text{Rf.}(\text{tra}_1, \dots, \text{tra}_{n-1}).\text{mar}) \end{aligned}$$

Notice that since  $\text{Tr.tra}$  is a partial function on markings, so is  $\text{Rf.pat}$ . If  $\text{Rf.pat.mar}$  is defined then  $\text{pat}$  is said to be a *symbolic execution* from  $\text{mar}$  to  $\text{Rf.pat.mar}$ . If  $\text{mar} = \text{Ini}$ , then  $\text{pat}$  is called an *initial execution*. By the *sequential symbolic behavior* of a Petri net, in symbols  $\text{Sbe.net}$  we mean the set of all initial executions of  $\text{net}$ :

$$\text{Sbe.net} = \{\text{pat} \mid \text{Rf.pat.Ini} = !\}$$

Notice that  $\text{Sbe.net}$  always includes empty path and is prefix closed. The latter property allows for a representation of infinitistic behaviors of nets by infinite branches (see Sec. 12.2.2) of initial executions

$$\text{pat}_1 \sqsubseteq \text{pat}_2 \sqsubseteq \dots$$

Two cases are of particular interest for further investigations. They correspond to *atomic nets*. A net is said to be *atomic* if it includes one place only. Atomic nets may be marked or unmarked. If such a net is marked then the initial marking of that net consists of its unique place. Otherwise we assume that the initial marking is empty. Now, consider two such nets on Fig. 12.3-1 where  $A$  and  $B$  represent any finite or empty sets of transitions.



Fig. 12.3-1 Two atomic nets

The sequential behaviors of these nets are the following languages:

$$\begin{aligned} (\text{BA})^*(\text{B} \mid \{\}) & \text{ — first net} \\ (\text{AB})^*(\text{A} \mid \{\}) & \text{ — second net} \end{aligned}$$

If  $A$  and  $B$  are not empty, both these nets represent loops which is due to the fact that the marking generated by  $B$  is empty, and, in turn, empty marking fires  $A$ . Due to the fact that both languages are prefix closed, they both represent not only finite but also infinite loopings.

Let's return now to the case of arbitrary nets, and observe that the ordering of transitions in their paths may have two causes:

1. it may be due to the sequential nature of executions; we simply have to put transitions one after another even if they can be executed independently (e.g. in parallel),
2. it may be due to the structure of the net which forces some transitions to be fired before some others.

To distinguish between these two causes in the description of net behaviors, we replace word languages by trace languages with a dependency relation between transitions defined in the following way:

$$(\text{tra}_1, \text{tra}_2) : \text{Dep.net} \quad \text{iff} \quad \text{Neigh.tra}_1 \cap \text{Neigh.tra}_2 \neq \{\}$$

As we see, two transitions are dependent on each other if one of the following conditions is satisfied:

- they share a common entry, i.e. they compete in taking a token from it,
- they share a common exit, i.e., they compete in putting a token into it,
- an exit place of the one is an entry place of the other, i.e., one of them waits for the firing of the other.

**Theorem 12.3.2-1** (see [76]) For any two initial paths  $\text{pat}_1, \text{pat}_2$  of  $\text{net}$ , if  $\text{pat}_1 \equiv_{\text{Dep.net}} \text{pat}_2$  then  $\text{Rf.pat}_1.\text{Ini} = \text{Rf.pat}_2.\text{Ini}$  ■

By the *concurrent symbolic behavior* of  $\text{net}$  we shall mean the following trace language generated from the sequential behavior of  $\text{net}$ :

$$\text{Cbe.net} = [\text{Sbe.net}]_{\text{Dep.net}}$$

Since in a net any independent transitions may be fired in an arbitrary order, the following fact is easy to see:

**Fact 12.3.2-1** For any net  $\text{net}$  its sequential behavior  $\text{Sbe.net}$  is  $\text{Dep.net}$ -complete, i.e.

$$\text{Sbe.net} = \text{T2L}([\text{Sbe.net}]_{\text{Dep.net}}).$$

If  $\text{Dep.net}$  is full, then the net is said to be *sequential*. In that case its concurrent behavior is a sequential trace language (Sec. 12.3.1).

**Fact 12.3.2-2** If  $\text{net}$  is sequential, then  $\text{Sbe.net}$  is the unique representant of  $\text{Cbe.net}$ .

Note that in a general case  $\text{Sbe.net}$  is the largest representant of  $\text{Cbe.net}$ .

Now, Petri nets may be given an executional semantics in the domain of trace languages. To do that A. Mazurkiewicz defines a universal operation of composition of two nets with disjoint sets of places. Note that the sets of transitions need not be disjoint<sup>108</sup>. Let

$$\text{net}_i = (\text{Place}_i, \text{Transition}_i, \text{Flow}_i, \text{Ini}_i) \quad \text{for } i = 1, 2$$

where  $\text{Place}_1 \cap \text{Place}_2 = \{\}$ . By *composition* of these two nets, in symbols  $\text{net}_1 + \text{net}_2$  we mean the net:

$$\text{net} = (\text{Place}, \text{Transition}, \text{Flow}, \text{Ini})$$

where

$$\begin{aligned} \text{Place} &= \text{Place}_1 \mid \text{Place}_2 \\ \text{Transition} &= \text{Transition}_1 \mid \text{Transition}_2 \\ \text{Flow} &= \text{Flow}_1 \mid \text{Flow}_2 \\ \text{Ini} &= \text{Ini}_1 \mid \text{Ini}_2 \end{aligned}$$

As is easy to see, the composition of nets is associative and commutative. Besides, the dependency relation of the resulting net is the union of dependency relations of component nets.

$$\text{Dep.net} = \text{Dep.net}_1 \mid \text{Dep.net}_2$$

The central theorem of Mazurkiewicz's approach to Petri nets is the following:

**Theorem 12.3.2-2**  $\text{Cbe}(\text{net}_1 + \text{net}_2) = \text{Cbe.net}_1 \parallel \text{Cbe.net}_2$  ■

A theorem which completes our sketch of Mazurkiewicz trace theory is about a decomposition of a net into a family of atomic nets. By an *atom* of  $\text{net}$  determined by a place  $\text{pla} : \text{Place}$  we mean a one-place net defined as follows:

$$\text{net}_{\text{pla}} = (\{\text{pla}\}, \text{Tra}_{\text{pla}}, \text{Flow}_{\text{pla}}, \text{Ini}_{\text{pla}})$$

where

$$\begin{aligned} \text{Tra}_{\text{pla}} &= \{\text{tra} \mid (\text{tra}, \text{pla}) : \text{Flow} \text{ or } (\text{pla}, \text{tra}) : \text{Flow}\} && \text{transitions adjacent to pla} \\ \text{Flow}_{\text{pla}} &= \{(\text{tra}, \text{pla}) \mid \text{pla} : \text{Exit.tra}\} \mid \{(\text{pla}, \text{tra}) \mid \text{pla} : \text{Entry.tra}\} && \text{edges including pla} \\ \text{Ini}_{\text{pla}} &= \text{Ini} \cap \{\text{pla}\} && \text{either } \{\text{pla}\} \text{ or empty set } \{\} \end{aligned}$$

<sup>108</sup> In fact, this is what makes the composition a non-trivial operation.

An atom of a Petri net consist of only one place plus all the edges that lead to or from this place including their transitions. Clearly

**Theorem 12.3.2-3** Every Petri net is a composition of its atoms. ■

As an immediate consequence of theorems 12.3.2-2 and 12.3.2-3 we may conclude

**Theorem 12.3.2-4** The concurrent behavior of a Petri net is a synchronization of concurrent behaviors of all its atoms. ■

### 12.3.3 Petri nets redefined

For the sake of building a denotational model of Petri nets we shall redefine the concepts of a net in a way equivalent to the former but more suitable for building an algebra of nets. In short, nets will be defined as finite sets of atoms.

Let **Identifier** be a set of identifiers over an alphabet including letters and digits. By an *atom* over **Identifier** we mean a 4-tuple which represents a marked or an unmarked net with one place:

$$\text{ato} : \text{Atom} = \text{Interface} \times \text{Place} \times \text{Marking} \times \text{Interface}$$

where

$$\begin{aligned} \text{int} & : \text{Interface} = \text{FinSet.Identifier} \\ \text{pla} & : \text{Place} = \text{Identifier} \\ \text{mar} & : \text{Marking} = \{0, 1\} \end{aligned}$$

We assume that both, places and transitions, are identifiers. In an atom (inp-int, pla, mar, out-int) the interfaces inp-int and out-int are called respectively *input interface* and *output interface*.

By an *abstract Petri net* over **Identifier** we mean any finite set of atoms with mutually different places. We call them “abstract nets” since in Sec. 12.3.4 we introduce concrete nets. Formally, they are, of course, different from nets of Sec. 12.3.2. We define two domains which will become carriers of the future *algebra of abstract nets*

$$\begin{aligned} \text{abn} & : \text{AbsNet} = \text{AbstractNet.Identifier} \\ \text{int} & : \text{Interface} = \text{FinSet.Identifier} \end{aligned}$$

where **AbstractNet** is a domain constructor analogously as  $\Rightarrow$  or  $\mapsto$ .

An abstract net **abn-1** is said to be a *subnet* of abstract net **abn-2** if  $\text{abn-1} \subseteq \text{abn-2}$ <sup>109</sup>. Abstract nets **abn-1** and **abn-2** are said to be *separated* if their sets of places and of transitions are disjoint. The constructors of the algebra of abstract nets are the following:

$$\begin{aligned} \text{build-empty-interface} & : & \mapsto \text{Interface} \\ \text{add-to-interface} & : \text{Identifier} \times \text{Interface} & \mapsto \text{Interface} \\ \text{build-unmarked-atom} & : \text{Interface} \times \text{Identifier} \times \text{Interface} & \mapsto \text{AbsNet} \\ \text{build-marked-atom} & : \text{Interface} \times \text{Identifier} \times \text{Interface} & \mapsto \text{AbsNet} \\ \text{assemble-abs-nets} & : \text{AbsNet} \times \text{AbsNet} & \mapsto \text{AbsNet} \end{aligned}$$

We skip obvious definitions of the first four constructors. To define the last one we introduce an auxiliary function of prefixing places of nets with digits ‘1’ and ‘2’:

$$\begin{aligned} \text{prefix} & : \text{AbsNet} \times \{‘1’, ‘2’\} \mapsto \text{AbsNet} \\ \text{prefix}((\text{inp}, \text{pla}, \text{mar}, \text{out}), p) & = (\text{inp}, p \bullet \text{pla}, \text{mar}, \text{out}) & \text{where } p : \{‘1’, ‘2’\} \\ \text{prefix}(\{\text{ato-1}, \dots, \text{ato-n}\}, p) & = \{\text{prefix}(\text{ato-1}, p), \dots, \text{prefix}(\text{ato-n}, p)\} \end{aligned}$$

The operation of *assemblage* of two abstract nets is defined in the following way:

<sup>109</sup> Note that this notion of a subnet may be different from such notions defined by other authors.

$$\text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2}) = \text{prefix.}(\text{abn-1}, '1') \mid \text{prefix.}(\text{abn-2}, '2')$$

Now, due to the Theorem 12.3.2-3, we can claim that for every Petri net as defined in Sec. 12.3.2 there exist an “equivalent” abstract net, and vice versa. By “equivalent nets” we mean that their corresponding graphs are isomorphic.

Having defined an algebra of abstract nets we have to define three functions which describe their symbolic behaviors:

$$\begin{array}{lll} \text{Sbe} : \text{AbstractNet.Identifier} & \mapsto \text{Lan.Identifier} & \text{sequential symbolic behavior} \\ \text{Cbe} : \text{AbstractNet.Identifier} & \mapsto \text{TraLan.Identifier} & \text{concurrent symbolic behavior} \\ \text{Dep} : \text{AbstractNet.Identifier} & \mapsto \text{FinSet.}(\text{Identifier} \times \text{Identifier}) & \text{dependency relation} \end{array}$$

We denote them by the same symbols as in Sec. 12.3.2, and, of course, they must correspond — in an obvious sense — to these functions. All three function will be defined by structural induction starting from two basic cases of atomic nets:

$$\text{ato} = (\text{inp}, \text{pla}, 0, \text{out}) \quad \text{and} \quad \text{ato} = (\text{inp}, \text{pla}, 1, \text{out}).$$

In this case (cf. Fig. 12.3-1):

$$\begin{array}{ll} \text{Sbe.ato} & = (\text{inp} \times \text{out})^{c^*} \times (\text{inp} \mid \{()\}) \\ \text{Dep.ato} & = (\text{inp} \mid \text{out}) \times (\text{inp} \mid \text{out}) \\ \text{Cbe.ato} & = [\text{Sbe.ato}]_{\text{Dep.ato}} \end{array} \quad \begin{array}{ll} \text{Sbe.ato} & = (\text{out} \times \text{inp})^{c^*} \times (\text{out} \mid \{()\})^{110} \\ \text{Dep.ato} & = (\text{inp} \mid \text{out}) \times (\text{inp} \mid \text{out}) \\ \text{Cbe.ato} & = [\text{Sbe.ato}]_{\text{Dep.ato}} \end{array}$$

In the case of atomic nets the dependency relations are full, and therefore their concurrent behaviors are sequential.

Note that a sequential behavior of a net is a set of sequence of identifiers representing transitions. Consequently it may be regarded as a language over an alphabet `Identifier`. Since identifiers are themselves sequences of letters, in the definition of `Sbe.ato` we have used Cartesian product and Cartesian star rather than concatenation and star of languages. Now, given two arbitrary abstract nets `abn-1` and `abn-2` and their composition

$$\text{abn} = \text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2})$$

we set (cf. Theorem 12.3.2-2)

$$\begin{array}{ll} \text{Cbe.abn} & = \text{Cbe.abn-1} \parallel \text{Cbe.abn-2} \\ \text{Sbe.abn} & = \text{T2L.}(\text{Cbe.abn}) \\ \text{Dep.abn} & = \text{Dep.abn-1} \mid \text{Dep.abn-2} \end{array}$$

We recall that by the definition of the synchronization operation  $\parallel$ , traces of `Cbe.abn` are built over `Dep.abn`. Note now that the following equations are satisfied

$$\begin{array}{ll} \text{Sbe.abn} = \text{T2L.}(\text{Cbe.abn-1} \parallel \text{Cbe.abn-2}) = & \text{by the definition of Cbe} \\ \text{T2L.}([\text{Sbe.abn-1}]_{\text{Dep.abn-1}} \parallel [\text{Sbe.abn-2}]_{\text{Dep.abn-2}}) = & \text{by Theorem 12.3.1-4} \\ \text{T2L.}([\text{Sbe.abn-1} \parallel \text{Sbe.abn-2}]_{\text{Dep.abn-1} \mid \text{Dep.abn-2}}) & \end{array}$$

This means that the sequential behavior of an assemblage of two nets is expressible as a combination of the sequential behaviors of the component nets.

**Fact 12.3.3-1** For any two abstract nets `abn-1` and `abn-2`

$$\text{Sbe.}(\text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2})) = \text{T2L.}([\text{Sbe.abn-1} \parallel \text{Sbe.abn-2}]_{\text{Dep.abn-1} \mid \text{Dep.abn-2}}).$$

It is worth recalling in this context that (cf. Fact 12.3.2-1) the equality:

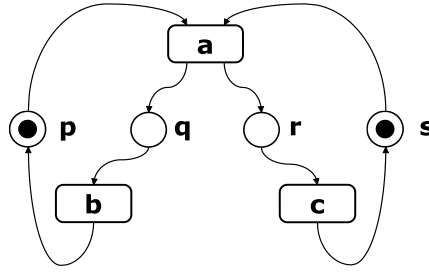
$$\begin{array}{l} \text{Cbe.}(\text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2})) = \\ [\text{Sbe.}(\text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2}))]_{\text{Dep.abn-1} \mid \text{Dep.abn-2}} \end{array}$$

<sup>110</sup> Here we half formally assume that Cartesian product is associative which means that  $(\text{inp} \times \text{out})^{c^*}$  and  $(\text{inp} \times \text{out})^{c^*}$  are set of sequences of identifiers, rather than set of sequences of pairs of identifiers.



and the fact that  $\text{Sbe.}(\text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2}))$  is  $\text{Dep.abn-1} | \text{Dep.abn-2}$  – complete.

Consider as an example an abstract net consisting of two interleaving loops in Fig. 12.3-2.



**Fig. 12.3-2 A net with two interleaving transitions**

This net may be regarded as an assemblage of four atomic nets, where  $a$  stands for  $\{(a)\}$ , and similarly for other transitions:

$$\begin{array}{lll} \text{abn-1} = (b, p, 1, a), & \text{Sbe.abn-1} = (ab)^* (a \mid ()), & \text{Dep.abn-1} = \{a, b\}^{c^2} \\ \text{abn-2} = (b, q, 0, a), & \text{Sbe.abn-2} = (ab)^* (a \mid ()), & \text{Dep.abn-2} = \{a, b\}^{c^2} \\ \text{abn-3} = (a, r, 0, c), & \text{Sbe.abn-3} = (ac)^* (a \mid ()), & \text{Dep.abn-3} = \{a, c\}^{c^2} \\ \text{abn-4} = (c, s, 1, a), & \text{Sbe.abn-4} = (ac)^* (a \mid ()), & \text{Dep.abn-4} = \{a, c\}^{c^2} \end{array}$$

Since the dependency relations of all these nets are full, they are sequential nets, and their concurrent behaviors are sequential trace languages. The same concerns the assembled nets

$$\begin{array}{ll} \text{abn-12} = \text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2}), & \text{Dep.abn-12} = \{a, b\}^{c^2} \\ \text{abn-34} = \text{assemble-abs-nets.}(\text{abn-3}, \text{abn-4}), & \text{Dep.abn-34} = \{a, c\}^{c^2} \end{array}$$

In this case

$$\begin{aligned} \text{Cbe.abn-12} &= [(ab)^* (a \mid ())] \parallel [(ab)^* (a \mid ())] = && \text{by Theorem 12.3.1-4} \\ &= [((ab)^* (a \mid ())) \parallel ((ab)^* (a \mid ())) ] = \\ &= [(ab)^* (a \mid ())] \end{aligned}$$

$$\text{Sbe.abn-12} = (ab)^* (a \mid ()) \quad \text{by Fact 12.3.2-2 since abn-12 is sequential}$$

Analogously

$$\begin{aligned} \text{Cbe.abn-34} &= [((ac)^* (a \mid ())) ] \\ \text{Sbe.abn-34} &= (ac)^* (a \mid ()) \end{aligned}$$

Assembling these two sequential nets we get a non-sequential net:

$$\begin{array}{ll} \text{abn} = \text{assemble-abs-nets.}(\text{abn-12}, \text{abn-34}) & \text{with } \text{Dep.abn} = \{a, b\}^{c^2} \mid \{a, c\}^{c^2}, \\ & \text{ind.Dep.abn} = \{(b, c), (c, b)\} \end{array}$$

For this net

$$\begin{aligned} \text{Cbe.abn} &= [(ab)^* (a \mid ())] \parallel [(ac)^* (a \mid ())] = \\ &= [((ab)^* (a \mid ())) \parallel ((ac)^* (a \mid ())) ] \\ \text{Sbe.abn} &= \text{T2L.} [((ab)^* (a \mid ())) \parallel ((ac)^* (a \mid ())) ] \end{aligned}$$

## 12.3.4 Petri nets with data flow

Petri nets defined so far describe control flows of concurrent programs. To enrich them with a mechanism of data flow we assign bundles of computations to transitions. Let **State** be an arbitrary set of states. At this stage of our investigations we do not need to assume anything about states. By *transition dictionaries* we mean mappings:

$$\text{tdi} : \text{TraDic} = \text{Identifier} \Rightarrow \text{Bun.State}$$

which assign bundles of states to transitions. By a *concrete net* we mean a pair consisting of an abstract net and a transition dictionary:

$\text{cne} : \text{ConNet} = \text{AbsNet} \times \text{TraDic}$

Concrete net  $(\text{abn-1}, \text{tdi-1})$  is called a *subnet* of a concrete net  $(\text{abn-2}, \text{tdi-2})$  if

$$\text{abn-1} \subseteq \text{abn-2} \text{ and } \text{tdi-1} \subseteq \text{tdi-2}.$$

A concrete net  $(\text{abn}, \text{tdi})$  is said to be *well formed* if all transitions of  $\text{abn}$  belong to the domain of  $\text{tdi}$ . By a *concrete execution* of a well-formed net  $(\text{abn}, \text{tdi})$  we mean a sequence of pairs

$$((\text{tra-1}, \text{com-1}), \dots, (\text{tra-n}, \text{com-n})) \quad \text{where } \text{com-i} : \text{State}^{c+}$$

such that

- (1)  $(\text{tra-1}, \dots, \text{tra-n}) : \text{Sbe.abn}$
- (2)  $\text{com-i} : \text{tdi.tra-i} \quad \text{for } i = 1; n$
- (3)  $\text{com-1} \bullet \dots \bullet \text{com-n} \neq ()$

Condition (1) expresses the fact that a concrete execution can happen symbolically, and condition (3) — that it can happen semantically.

We say that a concrete execution *starts in a set of states*  $\text{con} \subseteq \text{State}$  if the first state of  $\text{com-1}$  belongs to  $\text{con}$ . By

$$\text{coe} : \text{ConExe.cne} = \{((\text{tra-1}, \text{com-1}), \dots, (\text{tra-n}, \text{com-n})) \mid (1), (2), (3) \text{ satisfied}\}$$

we shall denote the set of all concrete executions of a well-formed concrete net  $\text{cne}$ . By a *denotational behavior* of a well-formed concrete net  $(\text{abn}, \text{tdi})$  we mean a bundle defined in the following way:

$$\begin{aligned} \text{Dbe.}(\text{abn}, \text{tdi}) = \{ & \text{com-1} \bullet \dots \bullet \text{com-n} \mid (\exists (\text{tra-1}, \dots, \text{tra-n}) : \text{Sbe.abn}) \\ & ((\text{tra-1}, \text{com-1}), \dots, (\text{tra-n}, \text{com-n})) : \text{ConExe.}(\text{abn}, \text{tdi}) \} \end{aligned}$$

We say that a concrete executions is a *prefix* of another one, in symbols

$$((\text{tra-1}, \text{com-11}), \dots, (\text{tra-1}, \text{com-1n})) \sqsubset ((\text{tra-21}, \text{com-21}), \dots, (\text{tra-2m}, \text{com-2m})) \quad (*)$$

iff

$$n < m \text{ and } ((\text{tra-1}, \text{com-11}), \dots, (\text{tra-1}, \text{com-1n})) = ((\text{tra-21}, \text{com-21}), \dots, (\text{tra-2n}, \text{com-2n}))$$

Of course,  $\text{ConExe.}(\text{abn}, \text{tdi})$ , similarly to  $\text{Sbe.abn}$  is prefix closed. Note that by the definition of a concrete execution if  $(*)$  holds then

$$\text{com-21} \bullet \dots \bullet \text{com-2m} \neq ()$$

and therefore

$$\text{com-11} \bullet \dots \bullet \text{com-1n} \sqsubset \text{com-21} \bullet \dots \bullet \text{com-2m} \text{ or}$$

$$\text{com-11} \bullet \dots \bullet \text{com-1n} = \text{com-21} \bullet \dots \bullet \text{com-2m}$$

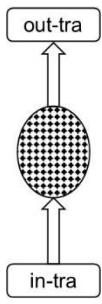
The equality may take place if all computations  $\text{com-2}(n+1), \dots, \text{com-2m}$  are appropriate one-element computations.

By a *concrete branch* of a concrete net we mean a finite or infinite sequence of concrete executions of this net such that each of them, except the last one, if it exists, is a prefix of the next one. We say that a concrete execution is *terminal* in a concrete net if it is not a prefix of any other concrete execution of this net.

By an *assemblage* of two concrete nets we mean a concrete net whose abstract component is an assemblage of abstract nets and the dictionary is an overwriting of dictionaries:

$$\text{assemble-con-nets} : \text{ConNet} \times \text{ConNet} \mapsto \text{ConNet}$$

$$\begin{aligned} \text{assemble-con-nets.}((\text{abn-1}, \text{tdi-2}), (\text{abn-2}, \text{tdi-2})) = \\ (\text{assemble-abs-nets.}(\text{abn-1}, \text{abn-2}), \text{tdi-1} \blacklozenge \text{tdi-2}). \end{aligned}$$



By a *cocoon*<sup>111</sup> we mean a tuple  $(in\text{-}tra, (abn, tdi), out\text{-}tra)$  such that

1.  $(abn, tdi)$  is a well-formed concrete net,
2.  $in\text{-}tra$  and  $out\text{-}tra$  are transitions of  $abn$ ,
3.  $in\text{-}tra$  has no entrance places (orphan) and  $out\text{-}tra$  has no exit places (widow) in  $abn$ ,
4.  $in\text{-}tra$  is the only orphan and  $out\text{-}tra$  is the only widow of  $abn$ ,
5. initial marking of  $abn$  is empty.

Note that by the properties 4. and 5. all sequential executions of  $abn$  start with  $in\text{-}tra$ .

By the *flow* of a cocoon  $con = (in\text{-}tra, (abn, tdi), out\text{-}tra)$  we mean a bundle generated by symbolic executions of concrete net  $(abn, tdi)$  between input and output transitions, i.e. all executions at all (!) which terminate in  $out\text{-}tra$ .

$Flow.(in\text{-}tra, abn, tdi, out\text{-}tra) =$

$$\{com\text{-}1 \bullet \dots \bullet com\text{-}n \mid (\exists (tra\text{-}1, \dots, tra\text{-}n) : Sbe.abn) \\ tra\text{-}n = out\text{-}tra \text{ and} \\ ((tra\text{-}1, com\text{-}1), \dots, (tra\text{-}n, com\text{-}n)) : ConExe.(abn, tdi)\}$$

Of course

$$Flow.(in\text{-}tra, abn, tdi, out\text{-}tra) \subseteq Dbe.(abn, tdi).$$

Let  $con\text{-}1, con\text{-}2 \subseteq State$ . A cocoon  $coc$  is said to be *strongly totally correct*<sup>112</sup> or simply *correct* wrt a precondition  $con\text{-}pr$  and a postcondition  $con\text{-}po$ , in symbols

**pre**  $con\text{-}pr$ ;  $coc$  **post**  $con\text{-}po$

if

1. there is no infinite concrete branch that starts with a state in  $con\text{-}pr$ ; no semantic livelock,
2. all terminal concrete execution that start with a state in  $con\text{-}pr$  terminate with transition  $out\text{-}tra$  in a state in  $con\text{-}po$ ; no semantic deadlock.

Of course, **pre**  $con\text{-}pr$ ;  $coc$  **post**  $con\text{-}po$  iff (see Sec. 12.2.3)

$$con\text{-}p \subseteq Flow.coc \blacksquare con\text{-}po$$

Note that the freeness of semantic livelock and deadlock of a cocoon does not imply that the corresponding abstract net is free of livelock or deadlock respectively.

Poproszę kolegów o przykład ???

Let a concrete net  $cne\text{-}1$  be a subnet of a concrete net  $cne\text{-}2$ . By a *projection* of a concrete executions  $coe$  of  $cne\text{-}2$  on a subnet  $cne\text{-}1$ , in symbols

$$pro.abn\text{-}1.coe$$

we mean the result of removing from  $coe$  all pairs  $(tra, com)$  such that  $tra$  is not a transition of  $cne\text{-}1$ . For a formal definition of a similar function see Sec. 12.3.1.

Consider two concrete nets  $cne\text{-}i = (abn\text{-}i, tdi\text{-}i)$  for  $i = 1, 2$ , two corresponding sets of states (conditions)  $con\text{-}i \subseteq State$  for  $i = 1, 2$ , and let

<sup>111</sup> We use this word since a cocoon may be said to be a thread with a skein in the middle.

<sup>112</sup> In the case of deterministic programs total correctness means that the (unique) execution of the program terminates. The clean total correctness means that the execution terminates without abortion, and the strong total correctness of nondeterministic programs — that all executions terminate without abortion.

$cne = assemble-con-net.(cne-1, cne-2)$

We say that these nets are *semantically independent* for corresponding conditions, if

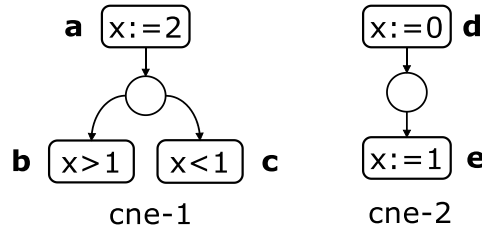
- (1)  $abn-1$  and  $abn-2$  are separated and
- (2) for any concrete execution of their composition

$coe : ConExe.(assemble-con-net.(cne-1, cne-2))$

if  $pro.abn-i.coe$  starts with a state in  $con-i$  then

$pro.abn-i.coe : ConExe.cne-i$  for  $i = 1, 2$ .

Condition (2) says that every projection of  $coe$  on  $abn-i$  that starts in  $con-i$  could happen as an independent execution of  $cne-i$ .



**Fig. 12.3-3 Two concrete dependent nets**

Consider as an example two concrete nets in Fig. 12.3-3, and assume their following properties:

1. both nets operate on states that are mappings from identifiers  $x, y$  to arbitrary numbers, i.e.  
 $sta : State = \{x, y\} \Rightarrow Number$
2. a common precondition  $con$  for both nets claims that variables  $x$  and  $y$  have been declared to be of type integer,
3. bundles assigned to transitions are the following sets of pairs of states (they are functional bundles):
  - a.  $\{(sta, sta[x/2]) \mid sta : State\}$
  - b.  $\{(sta, sta) \mid sta : State \text{ and } sta.x > 1\}$
  - c.  $\{(sta, sta) \mid sta : State \text{ and } sta.x < 1\}$
  - d.  $\{(sta, sta[x/0]) \mid sta : State\}$
  - e.  $\{(sta, sta[x/1]) \mid sta : State\}$

The following sequence is an example of a concrete execution of the assemblage of  $cne-1$  and  $cne-2$  where  $sta$  is an arbitrary state which satisfies  $con$ :

$(a, (sta, sta[x/2]), (d, (sta[x/2], sta[x/0]), (c, (sta[x/0], sta[x/0])), (e, (sta[x/0], sta[x/1])))$

The projection of this execution on  $cne-1$  is the following

$(a, (sta, sta[x/2]), (c, (sta[x/0], sta[x/0])))$

and it is not a concrete execution of  $net-1$  since

$(sta, sta[x/2]) \bullet ((sta[x/0], sta[x/0])) = ()$

If in transit  $d$  we set  $y:=0$  then our two concrete nets become semantically independent. Note that our nets become independent also if in  $d$  we set  $x:=2$ .

## 12.4 Building a language of concurrent programs

### 12.4.1 General assumptions about the language

In this section we propose a sketchy idea of how to extend a denotational model of a languages of sequential programming such as, e.g., **Lingua** or **Lingua-SQL**, to a denotational model covering concrete Petri nets. Similarly as in the case of **Lingua-SQL** we shall restrict our investigations to an algebra of denotations.

Let's assume at the beginning that **AlgDen** is a given algebra to be extended, and that the denotations of instructions, declarations and programs in this algebra are bundles of computations. The extended algebra **AlgDenCCP** is created from the former by adding to it the following new carriers:

|     |             |                                   |                         |
|-----|-------------|-----------------------------------|-------------------------|
| abn | : AbsNet    | = PetriNet.Identifier             | Petri nets              |
| int | : Interface | = FinSet.Identifier               | interfaces              |
| tdi | : TraDic    | = Identifier $\Rightarrow$ InsDen | transition dictionaries |
| cne | : ConNet    | = AbsNet x TraDic                 | concrete nets           |

and the corresponding constructors

|                        |                                      |                     |
|------------------------|--------------------------------------|---------------------|
| build-single-interface | : Identifier                         | $\mapsto$ Interface |
| add-to-interface       | : Identifier   Interface             | $\mapsto$ Interface |
| build-unmarked-atom    | : Interface x Identifier x Interface | $\mapsto$ AbsNet    |
| build-marked-atom      | : Interface x Identifier x Interface | $\mapsto$ AbsNet    |
| assemble-abs-nets      | : AbsNet x AbsNet                    | $\mapsto$ AbsNet    |
| build-dictionary       | : Identifier x TraDen                | $\mapsto$ TraDic    |
| add-to-dictionary      | : Identifier x TraDen x TraDic       | $\mapsto$ TraDic    |
| create-con-net         | : AbsNet x TraDic                    | $\mapsto$ ConNet    |
| asamble-con-nets       | : ConNet x ConNet                    | $\mapsto$ ConNet    |
| encapsulate-con-net    | : ConNet                             | $\mapsto$ InsDen    |

An instruction in the new algebra may be an instruction in the former sense or an *encapsulated concrete net*. In turn, transitions may carry arbitrary instructions.

The definitions of all new constructors but the last one are trivial. The last constructor given a concrete net returns a bundle of computations that belongs to the domain of instruction denotations:

$$\text{encapsulate-con-net}(\text{net}, \text{tdi}) = \text{Dbe}(\text{net}, \text{tid})$$

Note that this constructor “forgets”, in a sense, the structure of the argument net replacing it by a corresponding bundle. This bundle may appear later either as an argument of a constructor of instruction denotations in the basic algebra or a transition denotations assigned to a dictionary.

### 12.4.2 A case study of a structured constructor

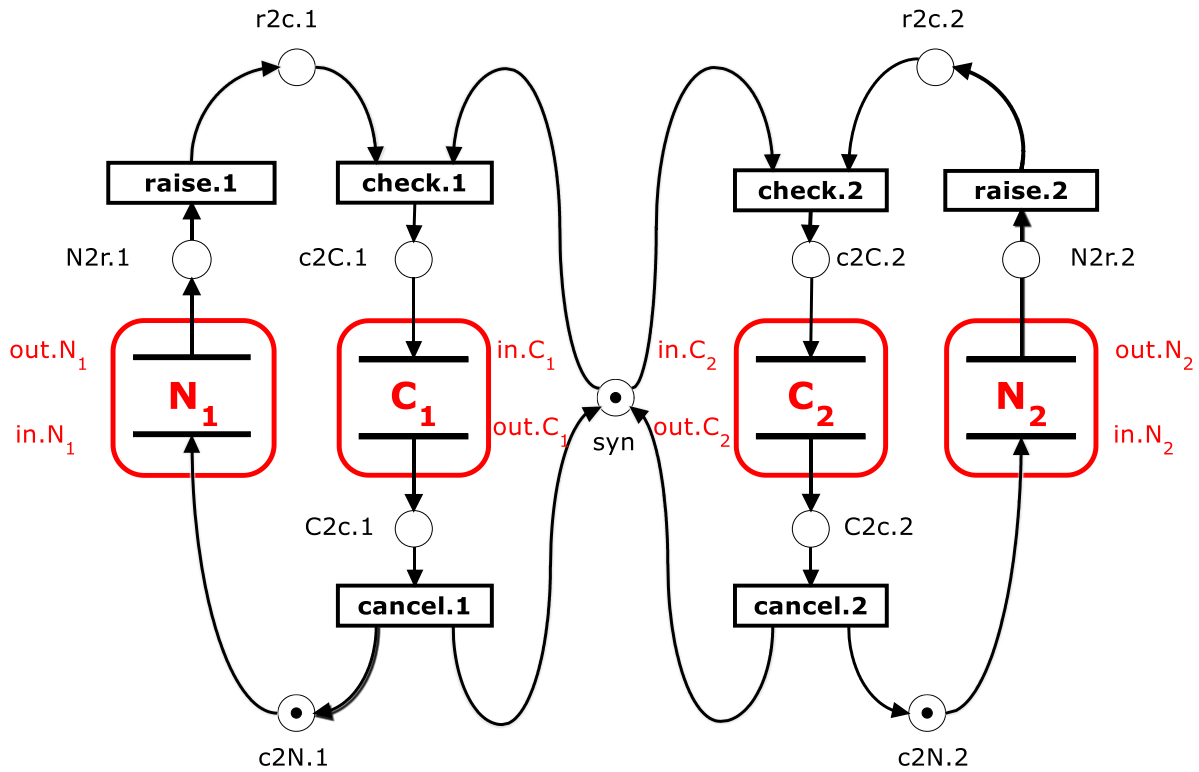
New constructors described in Sec. 12.4.1 allow to build concrete nets with arbitrary net structures. In the world of sequential programs this “flexibility” may be compared to allowing arbitrary flowchart programs built by **goto**'s. It is a well-known fact that such solution leads to programs that are hard to understand and even harder to prove correct.

A solution of this problem for sequential programming consists in restricting the control structures of instructions by structural constructors. For each such constructor we can create a dedicated proof rule, and in our case — a construction rule that guarantees correctness. In this way, instead of struggling with correctness proofs of “arbitrary programs” we are building a limited class of programs but their correctness proofs are implicit in the processes of their construction.

Here we propose an analogous solution for concurrent programming. However, we do not attempt to provide any “complete collection” of constructors for building nets since this would lead to a profound research certainly

going beyond the scope of our book. We shall limit our attention to only one of a structured constructor of concurrent programs that given two critical sections and two corresponding non-critical sections builds a net with Dijkstra's semaphores synchronizing the cooperation of sections. The expected correctness property of such a net, which we shall call *adequacy*, is described as follows:

1. the net is deadlock free,
2. each of its sections is livelock free,
3. critical sections are mutually excluded, i.e. their transitions can't interleave,
4. each non-critical section is excluded for its critical section,
5. after an execution of a non-critical section its corresponding critical section will be executed — a non-starvation property.



**Fig. 12.4-1 Critical sections with semaphores**

We start from building such a net in using constructors defined in Sec. 12.4.1. The structure of the abstract part of our target net is shown in Fig. 12.4-1. It consists of four subnets representing two critical sections, and two non-critical sections (all four in red) plus one synchronizing net (in black), that we shall call **SynNet**, including semaphores.

The target concrete net, let's call it **SemNet**, may be built as a composition of these five nets:

```
SemNet = assemble-con-nets.( SynNet,
    assemble-con-nets.( (N1, tdi-n1),
        assemble-con-nets.( (C1, tdi-com-1),
            assemble-con-nets.( (N2, tdi-n2), (C2, tdi-c2) ) ) ) ) )
```

Now, the question is, what do we have to assume about our five nets to make sure that **SemNet** is adequate. We start from assuming that the four sections constitute mutually separated cocoons:

```
non-crit.1 = (in.N1, (N1, tdi-nc.1), out.N1)
critical.1 = (in.C1, (C1, tdi-cr.1), out.C1)
non-crit.2 = (in.N2, (N2, tdi-nc.2), out.N2)
critical.2 = (in.C2, (C2, tdi-cr.2), out.C2)
```

Next we assume that the abstract part **AbsSynNet** of the synchronization net **SynNet** is a composition (a set) of the following atomic nets:

|  |   |
|--|---|
| (cancel.1, c2N <sub>1</sub> , 1, in.N <sub>1</sub> ) | (cancel.2, c2n.2, 1, int.N <sub>2</sub> ) |
| (out.N <sub>1</sub> , N2r.1, 0, raise.1)             | (out.N <sub>2</sub> , n2r.2, 0, raise.2)  |
| (raise.1, r2c.1, 0, check.1)                         | (raise.2, r2r.2, 0, check.2)              |
| (check.1, c2C.1, 0, in.C <sub>1</sub> )              | (check.2, r2c.2, 0, int.N <sub>2</sub> )  |
| (out.C <sub>1</sub> , C2c.1, 0, cancel.1)            | (out.C <sub>2</sub> , C2c.2, 0, cancel.2) |
| {cancel.1, cancel.2}, syn, 1, {check.1, check.2}     |   |

For simplicity we write **transition** for  $\{\text{transition}\}$ , and we give mnemotechnical names to places, e.g. c2N<sub>1</sub> is read as “from cancel to N<sub>1</sub>”.

To assure that **SynNet** realizes the mechanism of semaphores we have to assign an appropriate dictionary to this net. To do that we assume that **queue** is a variable which stores a FIFO queue of digits ‘1’ and ‘2’, and we define three typical operations on queues:

|  |              |  |
|--|--------------|--|
| first.(q <sub>1</sub> , ..., q <sub>n</sub> ) = q <sub>1</sub> ,                           | first.() = ? | (we assume that first is a partial function) |
| put.q.(q <sub>1</sub> , ..., q <sub>n</sub> ) = (q <sub>1</sub> , ..., q <sub>n</sub> , q) |              |  |
| cut.(q <sub>1</sub> , ..., q <sub>n</sub> ) = (q <sub>2</sub> , ..., q <sub>n</sub> ),     | cut.() = ()  |  |

The dictionary **tdi-sta-n** of **SynNet** is defined in the following way:

|                 |   |
|-----------------|---|
| tdi-sn.cancel.1 | = {(sta, sta[queue/cut.(sta.queue) ] }    |
| tdi-sn.raise.1  | = {(sta, sta[queue/put.'1'.(sta.queue)] } |
| tdi-sn.check.1  | = {(sta)   first.(sta.queue) = '1', }     |
| tdi-sn.cancel.2 | = {(sta, sta[queue/cut.(sta.queue) ] }    |
| tdi-sn.raise.2  | = {(sta, sta[queue/put.'2'.(sta.queue)] } |
| tdi-sn.check.2  | = {(sta)   first.(sta.queue) = '2', }     |

Hence **SynNet** = (**AbsSynNet**, **tdi-sn**). Now, we have to formulate assumptions sufficient for **SemNet** to be adequate. In the first place cocoons must be free of deadlocks and livelocks. Precisely speaking we have to assume that they are correct wrt some pre- and post-conditions. These conditions should also guarantee that semaphores will work as expected. This leads us to the following assumptions about **SemNet**:

There exist sets of states (pre- and post-conditions):

|                        |              |
|------------------------|--------------|
| nc-pre.i and nc-post.i | for i = 1,2, |
| cr-pre.i and cr-post.i | for i = 1,2  |

such that:

- pre** nc-pre.i; non-cri.i **post** nc-post.i for i = 1,2,
- pre** cr-pre.i; critical.i **post** cr-post.i for i = 1,2,
- for i ≠ j, non-cri.i is semantically independent of non-cri.j and critical.j under corresponding preconditions,
- for i ≠ j, critical.i is semantically independent of non-cri.j and critical.j under corresponding preconditions,
- cr-post.i • tdi-sn.cancel.i ⊆ nc-pre.i for i = 1,2
- nc-post.i • tdi-sn.raise.i • tdi-sn.check.i ⊆ cr-pre.i for i = 1,2
- cr-post.i • tdi-sn.cancel.i • tdi-sn.check.j ⊆ cr-pre.j for i ≠ j
- the value of **queue** is neither modified nor used by the instructions of cocoons.

Under these assumptions we can claim more about **SemNet** than just the freeness of deadlock, livelock and no-starvation. We can prove that for every concrete execution of **SemNet** that starts either from non-cri.1 in a state in nc-pre.1 or from non-cri.2 in a state in nc-pre.2, the following properties are satisfied:

- the executions is free of deadlock and livelock,
- critical sections **critical.1** and **critical.2** are mutually excluded, i.e. only one of them may be run at a time,
- sections **non-cr.i** and **critical.i** are mutually excluded for i = 1,2,



4. after the execution of **non-cr.i** an execution of **critical.i** will eventually happen for  $i = 1, 2$  (non-starvations),
5. after the execution of **non-cr.i** its next execution is possible only after the execution of **critical.i**, for  $i = 1, 2$ ,
6. after the execution of **critical.i** its next execution is possible only after the execution of **non-cr.i**, for  $i = 1, 2$ ,
7. transitions of **non-cr.i** may arbitrarily interleave with transitions of **critical.j** and **non-cr.j** for  $i \neq j$ .

At this point we have constructed a net with semaphores using just one constructor **assemble-con-net** but our construction does not meet, the following expectations of M. Ben-Ari (see [13] p. 145)

*(...) the semaphore is a low-level primitive because it is unstructured. If we were to build a large system using semaphores alone, the responsibility for the correct use of the semaphores would be diffused among all the implementors of the system. If one of them forgets to call **signal(S)** after a critical section, the program can deadlock and the cause of the failure will be difficult to isolate.*

In fact, our solution bases on the assumption that our net constructor receives **SynNet** as an argument which means that semaphores are built into **SemNet** by a programmer. What we would like to have is a constructor which given our four cocoons builds **SemNet**, which means that it builds **SynNet** which satisfies conditions from 5 to 8.

Here the critical condition is, of course, 8. To satisfy this condition we have to indicate an identifier **queue** with the assumed properties. The simplest constructive solution seems to be the selection of an identifier which is declared in states of **nc-pre.1** | **nc-pre.2** and does not appear syntactically in the instructions of our cocoons. Such a solution is, however, not feasible, since, informally speaking, the information carried by our cocoons is not sufficient to select such an identifier. The only solution that we can see at the moment is to assume that this identifier is given as an argument of the future constructor, i.e., is given by a programmer. In that case our expected constructor may be the following:

**semaphore-net**.(**non-cri.1**, **critical.1**, **non-cri.2**, **critical.2**, **queue**) =

**let**

(**in.N<sub>1</sub>**, (**N<sub>1</sub>**, **tdi-nc.1**), **out.N<sub>1</sub>**) = **non-cri.1**  
 (**in.C<sub>1</sub>**, (**C<sub>1</sub>**, **tdi-cr.1**), **out.C<sub>1</sub>**) = **critical.1**  
 (**in.N<sub>2</sub>**, (**N<sub>2</sub>**, **tdi-nc.2**), **out.N<sub>2</sub>**) = **non-cri.2**  
 (**in.C<sub>2</sub>**, (**C<sub>2</sub>**, **tdi-cr.2**), **out.C<sub>2</sub>**) = **critical.2**

**SynNet** = defined as above

**true**  $\rightarrow$  **assemble-con-nets**.( **SynNet**,  
     **assemble-con-nets**.( **N<sub>1</sub>**, **tdi-n<sub>1</sub>**),  
     **assemble-con-nets**.( **C<sub>1</sub>**, **tdi-com-1**),  
     **assemble-con-nets**.( **N<sub>2</sub>**, **tdi-n<sub>2</sub>**), (**C<sub>2</sub>**, **tdi-c<sub>2</sub>**) ) ) )

This definition looks similar to that of **SemNet** however now **SynNet** is not “given ahead”, but is built out of the arguments of **semaphore-net**.

Since **SynNet** is not an argument of our constructor, its elements should not appear in the prerequisites of our adequacy rule. Formally they do, but practically they don't, since if we assume that the pre- and post-conditions do not depend on variable **queue**, then 5., 6. and 7. may be reduced to inclusions (i.e. implications):

**cr-post.i**  $\subseteq$  **nc-pre.i**    for  $i = 1, 2$   
**nc-post.i**  $\subseteq$  **cr-pre.i**    for  $i = 1, 2$   
**cr-post.i**  $\subseteq$  **cr-pre.j**    for  $i \neq j$

Our last remark concerns the way we can view our adequacy rule. If we see it as a proof rule, it is rather poor, since its prerequisites are pretty strong. However, in our approach we aim at constructing correct programs rather than proving (arbitrary) programs correct. It practically means that before we apply our constructor we have to build its arguments in such a way that they have expected properties.



## 13 WHAT REMAINS TO BE DONE

Even though the book is already of a considerable volume, the majority of subjects have only been sketched. What remains to be done is enough for a few more books and also as a research and development area for many researchers and developers. Below we suggest a preliminary list of subjects which is certainly not complete. It considers both research problems as well as programming (implementational) tasks.

### 13.1 Theoretical problems to be solved

The order of problems is accidental.

1. Further development of our denotations model for concurrency, or a proposal of another model. In particular, more case studies of structured constructions (see Sec. 12.4.2).
2. The development of construction rules for recursive procedures. In particular, the development of rules for some typical situations where a procedure calls only itself (maybe more than once) and some more complicated situations where two or more procedures call themselves mutually.
3. The reintroduction of yokes to basic **Lingua**. In particular as a mechanism to describe specific properties of programs.
4. The development of some domain-specific languages dedicated to concrete applications in selected fields.
5. The completion of **Lingua-SQL** by more SQL mechanisms.
6. More studied on LL(k)-ness and LR(k)-ness of concrete and colloquial grammars, and how they depend on the transformations of grammars.

This list is, of course, not complete.

### 13.2 An ecosystem of programmers in Lingua-like languages

On a theoretical ground, our method of the development of correct metaprograms offers a collection of mathematical tools dedicated to developing and proving the correctness of metaprograms, i.e., theorems of the form:

**pre** prc : spr **post** poc.

It is fairly evident that the use of our tools in practice requires an assistance of a software system. Below we share some preliminary thoughts about such system. In our opinion such system should include (at least) the following modules:

1. an *intelligent editor* supporting the writing of metaprograms in **Lingua-V**; it may be implemented by building an extension in Visual Studio Code,
2. a *logistic engine* that would:
  - a. keep a repository of already-created correct metaprograms,
  - b. offer tools for building new correct metaprograms from programs developed earlier,
3. a *dedicated theorem-prover* supporting the application of program-construction rules,
4. an implementation of **Lingua**, i.e., a parser and an interpreter or compiler.

The editor should support two types of tasks:

1. keeping derived metaprograms syntactically correct,

2. keeping derived metaprograms statically correct<sup>113</sup>, e.g., no identifier should be declared twice in one program, or actual parameters of a procedure should be statically compatible with formal parameters.

The logistic engine should keep and update two repositories of correct metaprograms:

1. a global, possibly domain-specific, repository of metaprograms,
2. a local repository of correct metaprograms used in developing a “current program”, i.e., a program currently being developed.

The adding of a new metaprogram to a repository should be possible only by using one of the built-in program-construction rules (such as in Sec. 9.4 but probably also many other rules that should be added) and only under the condition that the associated theorem prover authorizes the application of this rule.

The dedicated theorem-prover should assist programmers in proving metaconditions appearing above the lines in program-construction rules — most frequently metaimplications, weak and strong equivalences of conditions or limited-replicability properties.

The theorem prover should keep and update two further repositories:

1. a repository of system-dependent theorems including:
  - a. basic theorems about data, values, types and denotations for a given **Lingua-V**,
  - b. currently proved program-construction rules,
2. a repository of program-dependent lemmas for the currently developed program including:
  - a. lemmas insuring that a given condition is perpetual in the current program,
  - b. lemmas insuring that a given condition is hereditary starting from an indicated cut of this programs.

Whereas part 1. of this repository will be modified only occasionally, part 2. will be updated practically in each step of program development. In each such step, given some earlier developed metaprograms we will create a new metaprogram by means of one of our construction rules. The application of a rule may require proving some “local lemmas”, e.g., some metaimplications of the form:

$\text{con1} \Rightarrow \text{con2}$ .

Such metaimplications will be always proved in the context of the current content of our repository, which formally means that our prover will prove the truth of the following metaformula:

$\text{con1} \Rightarrow \text{con2}$  **whenever** per-con **and** her-con

where

per-con    is the conjunction of all perpetual conditions from part 2.a of the library,  
 her-con    is the conjunction of all hereditary conditions from part 2.b of the library.

Similarly, whenever a programmer will develop a metaprogram of the form

**pre** prc:  
       spr  
**post** poc

the prover will elaborate

**pre** prc **and** per-con **and** her-con :  
       spr  
**post** poc **and** per-con **and** her-con

In each step of program development the part 2. of the library may be updated.

Regarding point 4. of our ecosystem, i.e., an implementation, this is a rather routine task that does not require any special comments. We should only mention that the first attempt to build an implementation of **Lingua**

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<sup>113</sup> This concept is related to so called “static semantics” that describes such properties of syntax that can’t be described by grammars and therefore can’t be checked by parsers.

(without objects) was undertaken by the students of Andrzej Blikle and Aleksy Schubert during a course given by them at the Department of Mathematics, Informatics, and Mechanics of Warsaw University in 2020 (see [38]). To tell the truth, the role of the first of us was limited to checking if the developed implementation was compatible with the model of **Lingua**. The bulk of the work was done by Aleksy and our students. The programming language of implementation was OCaml.

### 13.3 An ecosystem of Lingua-like language designers

The ecosystem supporting the design of programming languages in our framework might include the following components:

1. An intelligent editor supporting the writing of the definitions of the algebra of denotations, i.e.:
  - a. the definitions of the carriers of the algebra; here the editor might check if domain equations do not include an unacceptable recursion,
  - b. the definitions of the constructors of the algebra.
2. A system supporting syntax design:
  - a. a generator of an equational grammar of an abstract syntax from a given signature of an algebra of denotations,
  - b. a system supporting the development of an equational grammar of concrete syntax out of abstract syntax,
  - c. a system supporting the generation of an equational grammar of colloquial syntax out of concrete syntax,
  - d. a system supporting the generation of the definition of a restoring transformation.
3. A system supporting the writing of a definition of semantics.
4. A system supporting the development of a parser of the designed language out of the definition of its syntax.
5. A system supporting the development of an interpreter or compiler of the designed language out of the definition of its syntax and semantics.

Although the tasks of writing a programmer's interface and a language designer's interface are formally independent, one may think of two possible scenarios of building them:

1. first a version of **Lingua** is developed without using an ecosystem of a language designer and then the system is built using the developed **Lingua**,
2. a language-designer interface is written in one of the existing languages, and later our first practical **Lingua** is designed with a support of that system.

### 13.4 New standards of writing language reports and user manuals

#### 13.4.1 Two categories of reports

Denotational models should provide an opportunity to revise current practices in writing language reports and programmers' manuals. Let's start with reports.

Programming language **Lingua** has been thought as an example of a general-purpose programming language designed by using techniques offered by Denotational Engineering. It has a fully formalized denotational model consisting of three basic components:

1. a many-sorted algebra of denotations defined in a dedicated metalanguage MetaSoft,
2. a many-sorted algebra of syntax defined by means of an equational grammar,
3. a denotational semantics, that is a homomorphism from syntax to denotations.

In our approach, when a programming language is designed, language designer first defines successively (in this order) five algebras that belong to the denotational side of the model:

1. an algebra of data,
2. an algebra of data types,
3. an algebra of typed data,
4. an algebra of values,
5. an algebra of denotations.

In each of these algebras, the definitions of constructors refer to the constructors of previously defined algebras. The step from an algebra of values to an algebra of denotations requires defining a concept of a state.

Once an algebra of denotations is established three algebras of syntax are derived from its signature:

1. an algebra of abstract syntax,
2. an algebra of concrete syntax,
3. an algebra of colloquial syntax.

Colloquial syntax is “the ultimate” syntax to be used by programmers, whereas two former syntaxes are only some technical stages on a way to the later.

For abstract and concrete syntaxes there exist unique homomorphisms into the algebra of denotations. These homomorphisms constitute the semantics of these syntaxes.

For colloquial syntax there is, by principle, no homomorphism into denotations. There is, however, a *restoration morphism* that maps colloquial syntax into concrete syntax. A combination of the restoration morphism with the semantic of concrete syntax constitutes the semantics of colloquial syntax.

In our suggested approach we distinguish between two categories of reports of programming languages:

1. a report for language implementors; this report includes the definitions of the carriers and constructors of the five “denotational” algebras mentioned above plus the algebra of colloquial syntax and its semantics,
2. a report for programmers, that includes:
  - a. a definition of the algebra of colloquial syntax (by a grammar),
  - b. definitions of the carriers of the algebra of denotations,
  - c. a definition of semantics that constitutes a list of equations, called *semantic definitional clauses*, each assigning the denotation to one syntactic category; the constructors of the algebra of denotations are implicit in the semantic clauses.

Note that a report of the first category describes a hierarchy of algebras where in each of these algebras, except the first one, the definitions of constructors refer to the definitions in a preceding-level algebra. In turn, in a report dedicated to programmers the denotational algebras are not mentioned and their constructors are “hidden” in the definition of the semantic of the language.

## 13.4.2 Manuals for programmers

In our opinion a manual for programmers in a **Lingua**-like language should include a programmers’ report (Sec. 13.4.1) in the form of an annex plus a descriptive part where the following principles should be observed:

1. A manual should be addressed to professional programmers rather than to amateurs. The role of a manual is not to teach the skills of programming. Such textbooks are, of course, necessary, but they should teach the readers what programming is about, rather than the technicalities of a concrete language. The understanding of programming languages on one hand, and the science of programming on the other, are two separate skills. Manuals of building car engines do not teach physics and theoretical mechanics.
2. The meaning of programs should not be seen as recipes for executing orders by computers, but as mathematical operations that describe actions in a real world, e.g., shifting an amount of money from one bank

account to another one, or preventing that an airplane sits of an airport runway that is currently taken by another airplane.

3. Syntax should be perceived by programmers as a way to express the meanings of programs, and therefore, the meanings should come to the table first. I.e., we should first describe data, types, values and states and only then pass to the definitional clauses of semantics.
4. A manual should teach programmers not only how to express meanings by programs, but also how to ensure that the constructed programs are correct. Consequently it should include an exposition of sound program construction rules.

## 13.5 Programming experiments

For our idea of correct-program development to be noticed by the IT community, some convincing applications must be shown. In our (preliminary) opinion, an adequate field for such applications may be microprogramming because:

1. microprograms contain a relatively small number of the lines of code,
2. their correctness is highly critical,
3. highly critical is also the memory- and time-optimization of such programs.

## 13.6 Building a community of **Lingua** supporters

Our methods of designing programming languages and constructing programs may be assessed positively or negatively, but one thing seems evident — they are pretty far from current practices. The book offers a far-going change, and such changes always provoke groups of opponents and supporters. The former should be convinced, and the latter must be strengthened. And, of course, one has to start from the first task.

To realize that task, one has to give the potential supporters some — may be very simple — still sufficiently practical version of **Lingua**. An alternative may consist of encouraging them to build their own version. The first solution seems somewhat unrealistic since it would require finding an investor for a strange and utterly unknown product. The other way means that an experimental **Lingua** is built by volunteers and for volunteers, as in the case of Linux, Joomla!, or MySQL. However, such a product, although freely available, should not be open-source since this might lead to mathematically incorrect solutions and consequently to unsound program-construction rules.

Therefore, the **Lingua** builders community must elaborate rules of accepting new members and giving them the right to modify or introduce new sound rules of program construction.



## 14 INDICES AND GLOSSARIES

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### 14.3 Index of notations

|  |  |  |
|--|--|--|
| $()$ : empty word (tuple)<br>$\subseteq$ : a subset of<br>$\rightarrow$ : partial functions<br>$\mapsto$ : total functions<br>$\Rightarrow$ : mappings<br>$\bullet$ : composition of relations<br>$\odot$ : concatenation<br>$\exists$ : there exists<br>$\forall$ : for all | $\{\}$ : empty set/relation<br>$\sqsubseteq$ : partial order<br>$\Theta$ : pseudotype<br>$\{a.i \mid i=1;n\}$ : a set<br>$(a.i \mid i=1;n)$ : a tuple<br>$[a.i/b.i \mid i=1;n]$ : a mapping<br>$\text{Rel.}(A,B)$ : set of relations | $[A]$ : subset of identity rel.<br>$\blacklozenge$ : overwriting a function<br>$@$ : algorithmic formula<br>$\blacksquare$ : end of theorem/proof<br>$\Rightarrow$ : stronger than (metaimplication) |
|--|--|--|